Final Exam. Math 460, Topology.
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Name: ____________________
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Closed book. Closed Notes. 20 points per problem. Please write very legibly.

Do one problem from each section. Extra Credit problems do not carry any points; so do not spend any time on them unless you're sure you've done your best with the problems that do carry points.

Section I.

1. (a) Give the definitions of **path connected** and **simply connected**.
   (b) True or False: Suppose $f : X \to Y$ is continuous and onto. If X is path connected, then Y is path connected.
   (c) Let $f : X \to Y$ be a homeomorphism. Prove that if Y is simply connected, then every loop in X is null-homotopic. (Do not use the theorem that directly implies this.)

2. (a) True or False: Every continuous function $f : (0, 1) \to (0, 1)$ has a fixed point.
   (b) True or False: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose for some $a, b \in \mathbb{R}$ with $a < b$, $f([a, b]) \subseteq (a, b)$. Then f has a fixed point.

3. Is each of the following true or false? Prove your answers.
   (a) Every continuous map $f : S^1 \to S^1$ has a fixed point.
   (b) Let $f : S^1 \to \mathbb{R}$ be continuous. There exists a point $x \in S^1$ such that $f(x) = f(-x)$.

Section II.

4. Is each of the following true or false? Prove your answers.
   (a) Every subset of a compact topological space is compact.
   (b) If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact.

5. Is each of the following true or false? Prove your answers.
   (a) If A and B are connected subsets of a topological space X with $A \cap B \neq \emptyset$, then $A \cap B$ is connected.
   (b) If A and B are connected subsets of a topological space X with $A \cap B \neq \emptyset$, then $A \cup B$ is connected.

6. Using definitions only, prove that a subset of a metric space is open iff it is a union of open balls. (Hint: you will need to prove that a union of open sets is open, and every open ball is open.)

Section III.

7. (a) How many loops are there on the annulus $S^1 \times I$ such that no two of them are homotopic to each other as loops? Support your answer by precisely constructing the loops (but you don't need to prove that no two of them are homotopic to each other as loops).
   (b) How many loops are there on the torus $S^1 \times S^1$ such that no two of them are homotopic to each other as loops? Support your answer by drawing pictures, accompanied by an informal explanation (but you don't need to prove that no two of them are homotopic to each other as loops).

(c) How many loops are there on the projective plane $\mathbb{R}\mathrm{P}^2$ such that no two of them are homotopic to each other as loops? Support your answer by drawing pictures, accompanied by an informal explanation (but you don't need to prove that no two of them are homotopic to each other as loops).

8. Prove that the relation "f is isotopic to g" is an equivalence relation for the set of all homeomorphisms from a given topological space to another.

9. (a) Let $Y = \overline{B_5(0,0)} - B_1(0,0) \subset \mathbb{R}^2$. Let $h : I \to Y$ be given by $h(t) = (3,0) + (\cos(2\pi t), \sin(2\pi t))$, and $j : I \to Y$ be given by $j(t) = (-3,0) + (\cos(2\pi t), \sin(2\pi t))$. Prove that h and j are isotopic as simple loops in Y.

   (b) Let $f : I \to \mathbb{R}^2$ be given by $f(t) = (\cos(4\pi t), \sin(4\pi t))$. Let $g : I \to \mathbb{R}^2$ be given by $g(t) = (\cos(2\pi t), \sin(2\pi t))$. Find a homotopy between f and g to prove they are homotopic as loops in $\mathbb{R}^2$.

Section IV.

10. (a) State the classification theorem for closed surfaces.

    (b) Let P be the quotient space obtained by performing the following identifications on two "solid squares" $ABCD$ and $A'B'C'D'$ (i.e., including their insides, not just their boundaries): $AB \sim A'B'$, $CD \sim D'C'$. Describe P as best as you can. Explain your work.

    (c) Let Q be the quotient space obtained by performing the following identifications on the "solid hexagon" $ABCDEF$ (i.e., including the inside, not just the boundary): $AB \sim ED$, $BC \sim FE$, $CD \sim AF$. Describe Q as best as you can. Explain your work.

11. (a) Give the definition of the **discrete metric**.

    (b) Give the definition of the **taxicab metric**.

    (c) Give the definition of a **continuous** map from one metric space to another.

    (d) Give the definition of a **topology**.

    (e) Give the definition of **interior**.

    (f) Give the definition of the **product topology**.

    (g) Give the definition of a **manifold** (with boundary).

12. (a) Let $A = [0, 2)$ be given the subspace topology induced by the standard topology of $\mathbb{R}$. Is $[0, 1)$ open, closed, both, or neither, in A? Why?

    (b) State whether or not each of the following is a manifold. If it is, give its dimension and boundary. If it isn't a manifold, briefly explain your reasoning. In either case, state (without proof) whether or not it is compact.

    i. $\{(x, y) \in \mathbb{R}^2 \mid y \geq 0\} - \{(x, y) \in \mathbb{R}^2 \mid y = 0 \text{ and } x \in [2, 3)\}$.

    ii. $([0, 2] \cup [3, 5])/\{x \sim x + 3 \text{ for } x \in [0, 1)\}$.

    iii. $S^3 - B_{0.1}(1, 0, 0, 0)$.

Section V.

13. (a) Does there exist a homeomorphism from $[0, 1]$ to $\mathbb{R}$? If so, construct one (without any further proof). If not, give a brief explanation of your reasoning.

    (b) Does there exist a homeomorphism from $(0, 1)$ to $(0, \infty)$? If so, construct one (without any further proof). If not, give a brief explanation of your reasoning.

(c) Does there exist a homeomorphism from $[0, 1)$ to $\mathbb{R}$? If so, construct one (without any further proof). If not, give a brief explanation of your reasoning.

14. (a) Does there exist an equivalence relation $\sim$ on D^2 such that $D^2/\sim$ is homeomorphic to S^2? If so, state precisely what it is (without any further proof). If not, explain informally why not.

    (b) Does there exist an equivalence relation $\sim$ on S^2 such that $S^2/\sim$ is homeomorphic to the closed unit disk D^2? If so, state precisely what it is (without any further proof). If not, explain informally why not.

    (c) Does there exist an equivalence relation $\sim$ on $X = \mathbb{R}^3 - \{(0, 0, 0)\}$ such that $X/\sim$ is homeomorphic to $\mathbb{R}\mathrm{P}^2$? If so, state precisely what it is (without any further proof). If not, explain informally why not.

15. (a) Let P be the quotient space obtained by identifying the endpoints on the closed unit interval; i.e., $P = \{[0, 1]/\{0 \sim 1\}$. Which familiar topological space is Q homeomorphic to? Find a homeomorphism (without proof) to support your answer. Show your map is well-defined.

    (b) Let Q be the quotient space $\mathbb{R}^2/\{(x, y) \sim (x + 1, y); (x, y) \sim (x, y + 2)\}$. Which familiar topological space is Q homeomorphic to? Find a homeomorphism (without proof) to support your answer. Show your map is well-defined.

Extra Credit Problems

16. Can the floor of a room be so uneven that no matter where you put a four-legged table (or chair) on it, it will always wobble? Assume the floor's surface is continuous and the tips of the legs form a rectangle.

17. Suppose you have two pancakes of arbitrary shape next to each other on a tray. Prove that it is possible, with only one straight cut, to simultaneously divide each pancake into two equal halves (measured by area). (You may leave out tedious details, as long as your proof is otherwise correct and complete.)

18. Prove that the Möbius band M is not orientable, by giving a map $h : D^2 \to M$ and an isotopy between h and its mirror image.

19. (a) Represent $3T^2$ as a polygon with edges identified appropriately.

    (b) Represent $3\mathbb{R}\mathrm{P}^2$ as a hexagon with edges identified appropriately.