Suppose we were 2-dimensional creatures, living in $\mathbb{R}^2$. If a "2D Christopher Columbus" set out sail and traveled in a "constant direction", he would never come back home again. On the other hand, if the world were a closed surface (such as S^2 or T^2), then traveling in a "constant direction" might eventually get him back home.

Is the same possible for the 3D universe we live in? Is it possible that traveling in a fast spaceship along a "constant direction" for a long time might get us back to our starting point? *Can you think of any closed 3-manifolds?*

Definition 1. For $n \geq 0$, the n-**sphere** is defined as: $S^n = \{\vec{x} \in \mathbb{R}^{n+1} \mid d(\vec{x}, \vec{0}) = 1\}$.

Theorem 1. For $n \geq m$, S^n cannot be embedded in $\mathbb{R}^m$.

Example 1. According to the above theorem, can S^2 be embedded in $\mathbb{R}^2$? [1]

A "flatlander" (a 2D person living in "flatland", i.e., a 2D world) cannot really visualize a 2-sphere, since a 2-sphere can not be embedded in $\mathbb{R}^2$. Similarly, we, who live in a world that *locally* looks like $\mathbb{R}^3$, cannot visualize a 3-sphere, even though we might very well be living in one! But we can learn to work with it, and with many other things we cannot visualize, by learning from flatlanders!

Example 2. One of several ways a flatlander can think of a 2-sphere is: two closed disks glued along their boundaries. Similarly, we can think of a 3-sphere as two closed 3-balls (i.e., 3-dimensional balls in $\mathbb{R}^3$) glued along their boundaries.

Q: *Closed* has two meanings: one for topological subspaces, one for manifolds. Which one do we mean when we say *closed ball*? [2]

Q: How should $\sim$ be defined so that $S^3 \simeq [\overline{B_1(0,0,0)} \cup \overline{B_1(5,0,0)}]/\sim$? [3] We often write this as $\overline{B_1(0,0,0)} \cup_\partial \overline{B_1(5,0,0)}$. It means we're gluing the two balls along their boundaries.

Example 3. Let's try to see why the above description of S^3 is consistent with the formal definition given at the beginning. In other words, we'd like to find a homeomorphism between $S^3 = \{\vec{x} \in \mathbb{R}^4 \mid d(\vec{x}, \vec{0}) = 1\}$ and $\overline{B_1(0,0,0)} \cup_\partial \overline{B_1(5,0,0)}$.

Let's first do it in one dimension lower. Here's how a flatlander might describe a homeomorphism between $S^2 = \{\vec{x} \in \mathbb{R}^3 \mid d(\vec{x}, \vec{0}) = 1\}$ and $\overline{B_1(0,0)} \cup_\partial \overline{B_1(5,0)}$:

(1) Send the North Pole $(0,0,1) \in S^2 \subset \mathbb{R}^3$ to the point $(0,0) \in \overline{B_1(0,0)}$. (2) Send the South Pole $(0,0,-1) \in S^2 \subset \mathbb{R}^3$ to the point $(5,0) \in \overline{B_1(5,0)}$. (3) Send the Equator $\{(x,y,0) \in \mathbb{R}^3 \mid x^2+y^2 = 1\} \subset S^2 \subset \mathbb{R}^3$ to $\partial(\overline{B_1(0,0)}) = \partial(\overline{B_1(5,0)})$. (4) Send circles in the Northern Hemisphere parallel to the Equator to circles in $\overline{B_1(0,0)}$ centered at $(0,0)$. (5) Send circles in the Southern Hemisphere parallel to the Equator to circles in $\overline{B_1(5,0)}$ centered at $(5,0)$.

Q:What is the intersection of S^2 with the horizontal plane of height $1/2$ in $\mathbb{R}^3$? How about heights 0, 1, -1, 2? [4] These are called horizontal **cross sections** of S^2.

Q: How would you rigorously define a **horizontal hyperplane** in $\mathbb{R}^4$ (i.e., a horizontal $\mathbb{R}^3$ in $\mathbb{R}^4$)? [5]

Q:What is the intersection of $S^3\mathbb{R}^4$ with each of the following hyperplanes: (a) $\{(x,y,z,w) \in \mathbb{R}^4 \mid w = 0\}$. (b) $\{(x,y,z,w) \in \mathbb{R}^4 \mid w = 1\}$. (c) $\{(x,y,z,w) \in \mathbb{R}^4 \mid w = 1/2\}$. [6]

Q: Now try using horizontal cross sections of S^3 to show it is homeomorphic to $\overline{B_1(0,0,0)} \cup_\partial \overline{B_1(5,0,0)}$.

[1] No.
[2] We mean "topological"; i.e., a closed subset of $\mathbb{R}^3$.
[3] $\{(x,y,z) \sim (x+5,y,z)\}$.
[4] S^1, S^1, point, point, ϕ.
[5] $\{(x,y,z,w) \in \mathbb{R}^4 \mid w \text{ is a constant}\}$.
[6] A great 2-sphere. A point. A (not great) 2-sphere.

Definition 2. Let X be a topological space. An embedded circle in X is called a **simple closed curve** (scc). (*Simple* means not self-intersecting; *closed* means it's a loop – no endpoints.)

Definition 3. Let X be a topological space, with $A \subset B \subset X$. We say A **bounds** B iff $A = \partial B$.

Example 4. The unit circle $S^1 \subset \mathbb{R}^2$ bounds the unit disk $\overline{B_1(0,0)} \subset \mathbb{R}^2$.

Example 5. For 3D beings like us, it is easy to *see* that every scc C in the 2-sphere bounds a disk *on both sides*; i.e., there exist two embedded closed disks $D_1, D_2 \subset S^2$ with disjoint interiors such that $C = \partial D_1 = \partial D_2$. (Although "easy to see", this is rather difficult to prove rigorously. It's called the Jordan Curve Theorem.) However, for a flatlander, who thinks of S^2 as $\overline{B_1(0,0)} \cup_\partial \overline{B_1(5,0)}$, this is not as easy to see. Let $C \subset \overline{B_1(0,0)}$ be the circle of radius $1/4$ around the point $(1/2, 0)$. Draw a picture and shade in each of the two disks that C bounds in $\overline{B_1(0,0)} \cup_\partial \overline{B_1(5,0)}$.

Example 6. Now repeat the above example in one dimension higher: try to see why an embedded S^2 in S^3 bounds a closed 3-ball on *both* sides.

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Example 7. To work with $S^2 \times S^1$, it is often helpful to think of it as $S^2 \times I$ with $S^2 \times \{0\}$ glued to $S^2 \times \{1\}$.

Q: Is $S^2 \times S^1$ a manifold? If so, is it a closed manifold? If not, why not? [7]

Q: Find a 2-sphere in $S^2 \times S^1$ that does not bound a ball on either side. [8]

Q: Find a 2-sphere in $S^2 \times S^1$ that bounds a ball on one side only. [9]

Q: Is there a 2-sphere in $S^2 \times S^1$ that bounds a ball on both sides? [10]

Theorem 2. The only connected 3-manifold in which an embedded 2-sphere bounds a ball on both sides is the 3-sphere.

Proof: (Idea) If an embedded 2-sphere bounds a ball on both sides, then the two balls share the same boundary. But we already saw above that two balls glued along their boundaries yields an S^3.

Theorem 3. $S^3 \not\cong S^2 \times S^1$. Proof: Homework.

Example 8. Recall the definition of the connected sum of two n-manifolds, M and N: remove an open n-ball from each manifold; then $M - B_1$ and $N - B_2$ will each have a boundary component homeomorphic to S^{n-1}. Glue these boundaries together to obtain the connected sum of M and N.

Q: What is the connected sum of an arbitrary surface with a 2-sphere? Why? What is the connected sum of an arbitrary 3-mfd with a 3-sphere? Why? [11]

[7] It's a closed manifold.

[8] $S^2 \times \{x\}$, where x is any point in S^1.

[9] Take the boundary of any closed ball in $S^2 \times S^1$.

[10] No, by the next theorem.

[11] For both, the connected sum is homeomorphic to the original manifold.