In the previous section we saw the classification of all *closed* surfaces. The following theorem gives a classification of compact surfaces that are not closed (i.e., have boundary).

Theorem 1. Every compact surface with boundary is homeomorphic to some closed surface minus a finite set of disjoint open disks. In other words, every compact surface can be obtained from one of S^2, nT^2, or $n\mathbb{R}\mathrm{P}^2$ by removing finitely many disjoint open disks.

Sketch of Proof: Let F be a compact surface. Then its boundary consists of a number $m \geq 0$ of circles (why?). Take m disjoint copies $D_1, \cdots, D_m$ of the closed unit disk and glue each of them along its boundary to one of the m circle boundaries of F. The new surface F' we obtain is a closed surface; so, by the classification of closed surface, it is homeomorphic to S^2, nT^2, or $n\mathbb{R}\mathrm{P}^2$. So F equals S^2, nT^2, or $n\mathbb{R}\mathrm{P}^2$ minus a finite number of disjoint open disks.

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Orientable vs. non-orientable surfaces

Draw a clockwise oriented circle ↻ on a piece of paper. Can you isotop it to make it look like a counterclockwise oriented circle ↺ ? [1]

Now draw a ↻ on a Möbius band, and "push" it a full turn along the Möbius band. You must remember that the Möbius band is a 2-manifold, so ideally the paper has no thickness. So, as you're pushing the picture along, you should pretend the paper is see-through, and stop as soon as the picture comes back to its starting position, even if it may seem to be on the "other side" of the paper. Now stop reading and actually do this!

What did you get? The two copies look different! So, on the Möbius band, ↻ is isotopic to ↺ . Now do the same on a cylinder ($S^1 \times [0, 1]$). Is ↻ isotopic to ↺ on a cylinder?

In mathematics, we express these ideas by saying that the Möbius band is not orientable, while the plane and the cylinder are orientable.

Now, on a 2-sphere, there *is* a way to isotop a ↻ to a ↺ ! Can you see how? But, intuitively, we'd like to say that S^2 is orientable. So, instead of considering an oriented circle, we'll consider a closed disk whose circle boundary is oriented, i.e., a "filled-in" ↻ or ↺ . On S^2, a filled-in ↻ is not isotopic to a filled-in ↺ . Thus S^2 is orientable.

Informal definition: A 2-manifold M is *orientable* if a filled-in ↻ is not isotopic to a filled-in ↺ on M.

Is T^2 orientable ? How about an n-hole torus? How about the Klein bottle? [2]

The commonly used (formal) definition of orientability is too technical and would take up too much space here. Instead, below we have a non-standard but equivalent definition. It may not completely make sense until you see a precise definition of isotopy, later. So just read it, but for now you can work with the informal definition given above.

Definition 1. Let M be a 2-manifold, and let D^2 denote the closed unit disk in $\mathbb{R}^2$. Given an embedding $h : D^2 \to M$, the **mirror-image** $h' : D^2 \to M$ of h is defined by $h'(x, y) = h(-x, y)$. We say M is **non-orientable** if there is an embedding $h : D^2 \to M$ that is isotopic to its mirror-image. Otherwise, we say M is **orientable**.

Note. In the above definition, h and h' have the same image, i.e., $h(D^2) = h'(D^2)$. But they are different *maps*; for example, $h(1, 0) \neq h'(1, 0)$. When we talk about h and h' being isotopic, we are concerned not just with their images, but with the maps themselves. Again, this will make more sense when we see the formal definition of isotopy, later.

[1] No, it's impossible.

[2] Yes. Yes. No.

Theorem 2. For all $n \geq 1$, S^2 and nT^2 are orientable, and $n\mathbb{R}\mathrm{P}^2$ is non-orientable. Proof: Omitted.

Corollary 3. A closed surface is embeddable in $\mathbb{R}^3$ iff it is orientable. Proof: Homework.

Definition 2. For each $n \geq 1$, the surface nT^2 is said to have **genus** n, S^2 is said to have **genus** 0, and $n\mathbb{R}\mathrm{P}^2$ is said to have **genus** $n/2$.