Recall: How many connected 1-mfds are there? [1] It turns out that there are a lot more connected 2-mfds; in fact, there are infinitely many of them. Nevertheless, we can *classify* them, which roughly means we can systematically list them all, without any repetitions. (We'll soon have a better idea of what this means.) We will first concentrate only on 2-manifolds that are closed and can be embedded in $\mathbb{R}^3$; next, those that are closed but cannot be embedded in $\mathbb{R}^3$; and finally non-closed 2-mfds, but only compact ones. Non-compact 2-manifolds are more difficult to describe, and we'll skip them.

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Closed surfaces that are embeddable in $\mathbb{R}^3$

Example 1. What is the definition of a *closed* manifold? [2] Which surfaces have we seen so far that are closed and can be embedded in $\mathbb{R}^3$? Think before reading the following theorem!

Theorem 1. Every closed 2-manifold that can be embedded in $\mathbb{R}^3$ is homeomorphic to S^2 or to an n-hole torus (= the connected sum of n tori) for some $n \geq 1$. Proof: Omitted

Definition 1. Let (X, d) be a metric space, and let $A \subset X$. Given $\epsilon > 0$, the **ϵ-neighborhood** of A in X is defined as the set of all points in X whose distance is less than ϵ from some point in A: $N_\epsilon(A) = \{x \in X \mid d(x,a) < \epsilon \text{ for some } a \in A\}$.

Example 2. Let A be the horizontal line $y = 2$ in $\mathbb{R}^2$, and let $\epsilon = 0.1$. What is $N_\epsilon(A)$? What is $\partial(\overline{N_\epsilon(A)})$, the boundary of the closure of the ϵ-neighborhood of A? [3]

Example 3. Let $A = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq x,y,z \leq 1$ and at least two of x,y,z are in the set $\{0,1\}$ $\}$. Draw a picture of A. [4] Let $F = \partial(\overline{N_{0.1}(A)})$. Draw a picture of F. F is a closed surface $\subset \mathbb{R}^3$. So, according to the theorem above, F is homeomorphic to an n-hole torus for some n (since F is clearly not homeomorphic to S^2). You will find n in the homework assignment.

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Closed surfaces that are *not* embeddable in $\mathbb{R}^3$

Example 4. Let $X = [0,1]^2/\{(x,0) \sim (x,1)\ ,\ (0,y) \sim (1,y)\}$. Is X a 2-manifold? Is X embeddable in $\mathbb{R}^3$? [5]

In the following questions, you should be able to answer whether or not the given quotient space is a manifold. Determining whether or not it can be embedded in $\mathbb{R}^3$ is more difficult, and you may not be able to see why.

Let $M = [0,1]^2/\{(0,y) \sim (1,1-y)\}$. Is M a 2-manifold? Can you embed M in $\mathbb{R}^3$?

Let $K = [0,1]^2/\{(x,0) \sim (x,1)\ ,\ (0,y) \sim (1,1-y)\}$. Is K a 2-manifold? Can you embed K in $\mathbb{R}^3$?

Let $P = [0,1]^2/\{(x,0) \sim (1-x,1)\ ,\ (0,y) \sim (1,1-y)\}$. Is P a 2-manifold? Can you embed P in $\mathbb{R}^3$? [6]

Definition 2. $M = [0,1]^2/\{(0,y) \sim (1,1-y)\}$ is called the **Möbius band** (or Möbius strip). $K = [0,1]^2/\{(x,0) \sim (x,1)\ ,\ (0,y) \sim (1,1-y)\}$ is called the **Klein bottle**. $P = [0,1]^2/\{(x,0) \sim (1-x,1)\ ,\ (0,y) \sim (1,1-y)\}$ is called the **projective plane**, more commonly denoted by $\mathbb{R}\mathrm{P}^2$.

Remark. The projective plane is often also referred to as the *real projective plane.* This is in contrast to the *complex projective plane* $\mathbb{C}\mathrm{P}^2$, which we will not be studying.

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[1] Only four: S^1, $[a,b]$, $[a,b)$, (a,b).
[2] Compact, with no boundary.
[3] $N_\epsilon(A) = \{(x,y) \in \mathbb{R}^2 \mid 1.9 < y < 2.1\}$. $\partial(\overline{N_\epsilon(A)})$ consists of the two lines $y = 1.9$ and $y = 2.1$.
[4] A consists of the 12 edges of a cube.
[5] Yes to both; X is a torus.)
[6] M, K, and P are all 2-manifolds. Only M can be embedded in $\mathbb{R}^3$.

Theorem 2. The Projective Plane and the Klein Bottle cannot be embedded in $\mathbb{R}^3$. Proof omitted.

Theorem 3. (1) The boundary of a Möbius band is a circle: $\partial M \simeq S^1$. (2) Gluing a Möbius band and a closed disk along their circle-boundaries yields a projective plane: $M \cup_\partial \overline{D^2} \simeq \mathbb{R}\mathrm{P}^2$. (3) Gluing two Möbius bands along their circle-boundaries yields a Klein bottle: $M \cup_\partial M \simeq K$.

Proof. (Sketch)

(1) By definition, $M = [0,1]^2/\{(0,y) \sim (1,1-y)\}$. Therefore, ∂M consists of those points in $\partial([0,1]^2)$ that are *not* identified with any other point—except that the four corners of the square *are*, after being pairwise identified, in ∂M. So $\partial M = ([0,1] \times \{0\} \cup [0,1] \times \{1\})/\{(0,0) \sim (1,1)\ ,\ (1,0) \sim (0,1)\}$, which is homeomorphic to a circle.

(2) It's enough to show $M \simeq \mathbb{R}\mathrm{P}^2 - D^2$, as in the following diagrams.

(3) Homework. □

Theorem 4. Every closed 2-manifold that cannot be embedded in $\mathbb{R}^3$ is homeomorphic to the connected sum of n projective planes for some $n \geq 1$. Proof: Omitted.

Definition 3. For $n \geq 1$, the **n-hole torus** is the connected sum of n tori, denoted by nT^2. Similarly, the connected sum of n projective planes is denoted by $n\mathbb{R}\mathrm{P}^2$.

Corollary 5. Every closed 2-manifold is homeomorphic to either S^2 or nT^2 or $n\mathbb{R}\mathrm{P}^2$, for some $n \geq 1$.

Example 5. According to the above corollary, $T^2 \# \mathbb{R}\mathrm{P}^2$ is homeomorphic to either S^2 or nT^2 or $n\mathbb{R}\mathrm{P}^2$, for some n. Which is it? The next theorem answers this.

Theorem 6. $T^2 \# \mathbb{R}\mathrm{P}^2 \simeq 3\mathbb{R}\mathrm{P}^2$. Proof: Homework.

Corollary 7. $T^2 \# \mathbb{R}\mathrm{P}^2 \simeq \mathrm{K} \# \mathbb{R}\mathrm{P}^2$. Proof: Homework.

The above corollary may seem to suggest that $T^2 \simeq K$, which is not true!

Theorem 8. A torus is not homeomorphic to a Klein bottle. Proof: Homework.

Theorem 9. $S^2 \not\simeq \mathbb{R}\mathrm{P}^2$. Proof: Homework.

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You will need the following definition for the homework assignment.

Definition 4. Let A be a subset of a connected topological space X. To say A **separates** X means $X - A$ is not connected.

Example 6. Does S^1 separate $\mathbb{R}^2$? Does $[0, \infty)$ separate $\mathbb{R}^2$? Is there a non-separating embedded circle on the torus? How about a separating one? [7]

[7] Y, N, Y, Y. Draw pictures!
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