

There are lots of different topological spaces, some of which are very strange, counter-intuitive, and pathological, but also interesting. Some topological spaces, on the other hand, are in some sense very “nice and intuitive.” We are going to focus our studies on the latter kind: manifolds. These are in a way the ones that are the least strange and the most useful (or at least the most talked about). We will need two definitions before defining manifolds.

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**Definition 1.** Let  $x$  be a point in a topological space  $X$ . Any open subset of  $X$  that contains  $x$  is called a **neighborhood** of  $x$ .

(In some books, a neighborhood of  $x$  is defined as any set that contains an open set that contains the point  $x$ . Such books would then use the term *open neighborhood* for what we simply call above neighborhood. There are advantages and disadvantages to both approaches.)<sup>2</sup>

**Example 1.** Let  $X = [0, 1] \subset \mathbb{R}$ . Determine whether each of the following is a neighborhood of the point 0.3 in  $X$  (be careful: it says “in  $X$ ” not “in  $\mathbb{R}$ ”).

$(0.2, 0.8)$ ;  $[0, 1]$ ;  $[0, .7)$ ;  $(0.2, 0.4) \cup (0.6, 0.7)$ ;  $[0.2, 0.4)$ .<sup>1</sup>

**Example 2.** Let  $X = S^1 \subset \mathbb{R}^2$ . Give an example (other than  $S^1$  itself) of a neighborhood of the point  $(1, 0) \in S^1$ . Ans:  $\{(x, y) \in S^1 \mid -1/2 < y < 1/2\}$ .

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Definition 2. A topological space X is said to be **locally homeomorphic** to a topological space Y iff every point in X has some neighborhood that is homeomorphic to Y .

Example 3. Let $X = S^1 \subset \mathbb{R}^2$, $Y = (0, 1) \subset \mathbb{R}$.

Q: Is X locally homeomorphic to Y ? Is X homeomorphic to Y ?²

Q: Let $x \in X$. Is *every* neighborhood of x homeomorphic to Y ?³

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**Definition 3.** An  **$n$ -dimensional manifold** (or an  $n$ -manifold, for short) is a topological space that is Hausdorff, Second Countable, and locally homeomorphic to an open ball in  $\mathbb{R}^n$ .

For the time being, you should ignore Hausdorff and Second Countable. The important part of the definition that we’ll need to understand is: locally homeomorphic to an open ball in  $\mathbb{R}^n$ .

**Example 4.** Is  $S^1$  a 1-manifold?<sup>4</sup>

**Example 5.** Is a torus  $S^1 \times S^1$  a manifold?<sup>5</sup>

**Note.** The torus is often denoted as  $T^2$ .

**Example 6.** Let  $X = x\text{-axis} \cup y\text{-axis}$ . Give  $X \subset \mathbb{R}^2$  the subspace topology.

Q: Is  $X$  a manifold?<sup>6</sup>

Q: Is  $X - \{(0, 0)\}$  a manifold? Yes. Of what dimension? 1. Is it a connected manifold? No; it has four *components* (we’ll see a precise definition of this word later).

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<sup>1</sup>Yes. Yes!, why? Yes, why? Yes. No.

<sup>2</sup>Yes; why?. No; proved in a previous section.

<sup>3</sup>No; but every point  $x$  has *some* neighborhood that is homeomorphic to  $Y$ .

<sup>4</sup>Yes, because it is locally homeomorphic to an open ball in  $\mathbb{R}$ .

<sup>5</sup>Yes; every point has a neighborhood homeomorphic to an open ball (open disk) in  $\mathbb{R}^2$ . So it’s a 2-dimensional manifold.

<sup>6</sup>No: the origin has no neighborhood in  $X$  that’s homeomorphic to an open ball.

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Note. Is $(0, 1) \simeq \mathbb{R}$? ⁷ Is an open ball in \mathbb{R}^n homeomorphic to \mathbb{R}^n ? ⁸

Therefore, in the definition of manifold, instead of saying “... homeomorphic to an open ball in \mathbb{R}^n ,” we can (and often do) just say “... homeomorphic to \mathbb{R}^n .”

Example 7. Every point in S^1 has a neighborhood that is homeomorphic to \mathbb{R} , so S^1 is a 1-manifold.

Example 8. Is $(0, 1) \subset \mathbb{R}$ a manifold? Yes, it’s a 1-manifold. How about $[0, 1] \subset \mathbb{R}$? No, because the endpoints, 0 and 1, do not have neighborhoods in $[0, 1]$ that are homeomorphic to an open ball. But $[0, 1]$ is a manifold-with-boundary. We’ll see a precise definition for this next time.

⁷Yes.

⁸Yes.