

So far we know of one way to create new topological spaces from known ones: Subspaces. Now we will learn two other methods: 1. Product Spaces; and 2. Quotient Spaces.

Product Spaces

Recall: Given arbitrary sets X, Y , their product is defined as $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$.

Example 1. Let $X = [0, 1]$, $Y = [0, 1]$. Then $X \times Y$ is called the **closed unit square**. Draw a picture of this in $\mathbb{R}^2$ for yourself.

Example 2. Let $X = \mathbb{R}$, $Y = \mathbb{R}$. Then $X \times Y$ is the plane $\mathbb{R}^2$.

Example 3. Is every subset of $X \times Y$ of the form $A \times B$, where $A \subseteq X$ and $B \subseteq Y$? ¹

Definition 1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Their **product** is defined by: A set $U \subseteq X \times Y$ is open iff it is a (finite or infinite) union of sets of the form $A \times B$, where A is open in X and B is open in Y . We write: $(X \times Y, \mathcal{T})$, where $\mathcal{T} = \{\bigcup A_\alpha \times B_\alpha \mid A_\alpha \in \mathcal{T}_X, B_\alpha \in \mathcal{T}_Y\}$.

Example 4. Let $X = [0, 1]$, $Y = [0, 1]$. What are the open sets in $X \times Y$? Ans: They are unions of “open rectangles.”

Example 5. In the above definition, is $\mathcal{T}$ a topology on $X \times Y$?

1. Is $\phi \in \mathcal{T}$? Is $X \times Y \in \mathcal{T}$? ²
2. Is $\mathcal{T}$ closed under arbitrary unions? ³
3. Is $\mathcal{T}$ closed under finite intersections? ⁴

Example 6. What is $S^1 \times [0, 1]$? You can visualize it as a soda can without the top and bottom: a **cylinder**. You can also visualize it as the subset of $\mathbb{R}^2$ between two concentric circles, like a washer; the mathematical term for it is an **annulus**. Thus, annulus and cylinder are the same thing, i.e., homeomorphic — they’re just different ways of visualizing the same topological object.

Note. How is the unit circle S^1 a topological space? It is a subspace of $\mathbb{R}^2$.

What is $S^1 \times [0, 2]$? It too is a cylinder. Is this cylinder homeomorphic to the one above? ⁵

Example 7. What is $S^1 \times S^1$? A **torus**.

Example 8. Let $X = [0, 1] \times [0, 1]$. We can put a topology on X in two different ways:

First way: 1. Start with $\mathbb{R}$, with the Euclidean metric. 2. The metric induces a topology on $\mathbb{R}$. 3. Then $[0, 1]$ inherits the subspace topology from $\mathbb{R}$. 4. Then $[0, 1] \times [0, 1]$ is given the product topology.

Second way: $X = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1\} \subset \mathbb{R}^2$; so let X inherit the subspace topology from $\mathbb{R}^2$.

Q: Do these two methods produce the same topology on X ? Ans: Yes. Proof: Homework.

We sometimes abbreviate $[0, 1] \times [0, 1]$ by $[0, 1]^2$.

¹No. Find a counterexample.

²Yes to both; why?

³Yes; why?

⁴Yes (Extra Credit).

⁵Yes.

Quotient Spaces (also called Identification Spaces)

If we glue (connect) the two endpoints of a string together, we get a loop. If we “glue” or “identify” the two endpoints of $[0, 1]$ into one point, intuitively, we get a circle. If we take two closed disks and glue or identify their two circle boundaries together, we get a sphere S^2 . If we glue two opposite edges of a closed square, we get a cylinder. If we pairwise glue all opposite edges of a closed square, we get a torus. How can we make these ideas precise? We define quotient spaces (or identification spaces).

Recall: Let X be an arbitrary set. An **equivalence relation** on X is a relation (i.e., a set of ordered pairs) on X that is: **reflexive** ($x \sim x$), **symmetric** (if $x \sim y$, then $y \sim x$), and **transitive** (if $x \sim y$ and $y \sim z$, then $x \sim z$).

For each $x \in X$, the equivalence class of x is defined as: $[x] = \{y \in X \mid y \sim x\}$.

~~~~~

Informally speaking, if we take  $[0, 1]$  and “glue” 0 to 1, we get a circle. We represent the “gluing” by writing  $0 \sim 1$ .

**Definition 2.** Let  $X$  be a set,  $\sim$  an equivalence relation on  $X$ , and  $Q$  the set of all equivalence classes of  $\sim$ , i.e.  $Q = \{[x] \mid x \in X\}$ . Define a map  $q : X \rightarrow Q$  by: for each  $x \in X$ ,  $q(x) = [x]$ . The map  $q$  is called the **quotient map** (or the *identification map*) from  $X$  to  $Q$ . We sometimes write  $X/\sim$  instead of  $Q$ .

**Definition 3.** Let  $(X, \mathcal{T}_X)$  be a topological space,  $\sim$  an equivalence relation on  $X$ , and  $q : X \rightarrow Q$  the corresponding quotient map. The **quotient topology** on  $Q$  is defined as  $\mathcal{T}_Q = \{U \subseteq Q \mid q^{-1}(U) \in \mathcal{T}_X\}$ . In other words,  $U$  is declared to be open in  $Q$  iff its preimage  $q^{-1}(U)$  is open in  $X$ . The pair  $(Q, \mathcal{T}_Q)$  is called the **quotient space** (or the *identification space*) obtained from  $(X, \mathcal{T}_X)$  and the equivalence relation  $\sim$ .

**Example 9.** Let  $X = [0, 1]$ . Let  $\sim$  be the equivalence relation such that  $0 \sim 1$ , with no other distinct points related to each other. We want to understand what the points and the open sets in  $Q$  look like. There is some notation to get used to.

Q: Write the equivalence relation  $\sim$  as a set of ordered pairs. <sup>6</sup>

Q: Write  $[0]$  as a set. <sup>7</sup>

Q: Write  $[1]$  as a set. <sup>8</sup>

Q: Write  $[1/2]$  as a set. Ans:  $[1/2] = \{1/2\}$ .

Q:  $q^{-1}([0]) = ?$  Ans:  $\{0, 1\}$ .  $q^{-1}([1/2]) = ?$  Ans:  $\{1/2\}$ .

Q: Let  $A = [0, 1/2)$ . Is  $A$  open in  $X$ ? Yes; why? Let  $A' = q(A)$ . Is  $A'$  open in  $Q$ ? Ans: By definition,  $A'$  is open in  $Q$  iff its preimage  $q^{-1}(A')$  is open in  $X$ . What is  $q^{-1}(A')$ ? Be careful: it's not  $A$ !  $q^{-1}(A') = A \cup \{1\}$ ; why? So  $q^{-1}(A')$  is not open in  $X$ ; therefore  $A'$  is not open in  $Q$ .

Q: Let  $B = (0, 1/2)$ . Is  $A$  open in  $X$ ? Yes; why? Let  $B' = q(B)$ . Is  $B'$  open in  $Q$ ? Ans: Yes, b/c  $q^{-1}(B') = B$ , which is open in  $X$ .

Q: Find an open subset of  $Q$  that contains the point  $[0]$ . Ans: One trivial answer is  $Q$  itself; why?

Q: Find a *proper* open subset of  $Q$  that contains the point  $[0]$ . (Think before reading the answer.) <sup>9</sup>

~~~~~

Example 10. Find a homeomorphism from $[0, 1]/\{0 \sim 1\}$ to S^1 (just find a map, without proving it is a homeomorphism).

Ans: In polar coordinates: $f(x) = (1, 2\pi x)$. In rectangular coordinates: $f(x) = (\cos(2\pi x), \sin(2\pi x))$.

⁶ $\{(x, y) \in [0, 1]^2 \mid x = y \text{ or } (x = 0 \text{ and } y = 1) \text{ or } (x = 1 \text{ and } y = 0)\}$.

⁷By definition, $[0] = \{x \in X \mid x \sim 0\} = \{0, 1\}$.

⁸ $[1] = [0] = \{0, 1\}$.

⁹Let $C = [0, 1/3) \cup (2/3, 1]$. Then $q(C)$ contains $[0]$; why? And $q(C)$ is open in Q ; why?

Example 11. Let X be the unit closed square, $[0, 1]^2$. Identify the left edge of X with its right edge: $Y = X / \{(0, b) \sim (1, b) \mid b \in [0, 1]\}$.

Q: Is X a subset of $\mathbb{R}^2$? Yes. Is Y a subset of $\mathbb{R}^2$? No. Is Y a subset of $\mathbb{R}^3$? No.

Y is an abstract set, with the quotient topology. But Y can be shown to be homeomorphic to the cylinder $S^1 \times [0, 1] \subset \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$. So we say Y is a cylinder, even though it's not really a subset of $\mathbb{R}^3$.

Example 12. What equivalence relation on $X = [0, 1]^2$ gives a torus as the quotient space? ¹⁰

¹⁰ $[0, 1]^2 / \{(a, 0) \sim (a, 1), (0, b) \sim (1, b), \}$.