The Euclidean Real Line

Recall Calculus (no metric spaces): On the real line $\mathbb{R}$ the distance between two points x and y is given by $|x-y|$. Then we define continuity: $|x-a| < \delta$ implies $|f(x) - f(a)| < \epsilon$. Then we prove that continuous functions have certain (nice) properties, such as: the sum of two continuous functions is continuous; the composition of two continuous functions is continuous; etc. We notice that for our proofs to work we don't exactly have to have the absolute value function in the definition of continuity: the same proof would work if instead of the absolute value function we had any function with the following three properties:

1. $|x-y| = 0$ iff $x = y$.
2. $|x-y| = |y-x|$.
3. $|x-z| \leq |x-y| + |y-z|$.

So that gives us the following idea:

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Generalize to Metric Spaces

We abstract the three properties of the absolute value:

A metric space is a set X with $d : X \times X \to [0, \infty)$ satisfying
1. $d(x, y) = 0$ iff $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq d(x, y) + d(y, z)$.

Then we generalize the definition of continuity: $d(x, a) < \delta$ implies $d(f(x), f(a)) < \epsilon$ (use d instead of absolute value).

A few more definitions to imitate and generalize our familiar Euclidean world:

Ball: $B_r(x) = \{y \in X \mid d(x, y) < r\}$ = the set of all points whose distance from x is less than r .

Open set A (to imitate open intervals in $\mathbb{R}$): $\forall x \in A$, $\exists r$ s.t. $B_r(x) \subseteq A$.

We'd like to generalize even further: talk about open and closed sets and continuous functions without having a distance function (metric) d at all! Why do we want to do away with a distance function? Because in topology we deform objects, which means distances change. So we're not really concerned with distance; we want to know which properties of an object remain the same when it's deformed. Perhaps surprisingly, it is much easier to talk about open and closed sets, continuity, and many other things, without talking about a distance function! These comments will become more clear shortly.

We notice an important fact about metric spaces:

Theorem 1. Given metric spaces M and N, a function $f : M \to N$ is continuous iff the preimage (or inverse image) of every open set is open; i.e., f is continuous iff for every open subset $U \subseteq N$, $f^{-1}(U)$ is an open subset of M.

Proof: Homework.

This characterization of continuity avoids any mention of a distance function; instead it only talks about open sets. But to define *open*, we still need a distance function, right? No, as we'll see shortly!

Review of important properties of open sets in a metric space X :
1. ϕ and X are open.
2. Union of open sets is open.
3. Finite intersection of open sets is open.

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Generalize to Topological Spaces

A topology on X is a *declaration* of which subsets of X we are *choosing* to call open; we can choose any collection of subsets we desire, as long as the three conditions listed above are satisfied.

Definition 1. A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X satisfying:

1. ϕ and X are in $\mathcal{T}$.
2. The union of any collection of sets in $\mathcal{T}$ is in $\mathcal{T}$.
3. The intersection of any finite collection of sets in $\mathcal{T}$ is in $\mathcal{T}$.

The last two conditions are often stated as: $\mathcal{T}$ is **closed under** unions and finite intersections.

The pair $(X, \mathcal{T})$ is called a **topological space**. The elements of $\mathcal{T}$ are called **open** sets. A subset $A \subseteq X$ is **closed** iff its complement in X is open.

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Example 1. Let $X = \mathbb{R}$, $\mathcal{T} = \{X, \phi, (1,5), (4,6)\}$. Is $\mathcal{T}$ a topology on X? [1]

Example 2. Let $X = \mathbb{R}$, $\mathcal{T} = \{X, \phi, [1,5], (4,6)\}, (4,5], [1,6)\}$. Is $\mathcal{T}$ a topology on X? [2]

Example 3. Let $X = \mathbb{R}$, $\mathcal{T} = \{(a,b) \mid a, b \in \mathbb{R}\} \cup \{\mathbb{R}, \phi\}$. Is $\mathcal{T}$ a topology on X? [3]

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Theorem 2. Every metric space is a topological space. More precisely, given a metric (i.e., a distance function) d on a set X, let $\mathcal{T}$ be the collection of subsets of X that are open according to d. Then $\mathcal{T}$ satisfies the definition of being a topology.

Proof. Let (X, d) be a metric space, and let $\mathcal{T}$ be the collection of subsets of X that are open according to d. Then, by our previous work (HW), we know that $\mathcal{T}$ satisfies the three conditions in the definition of a topology. Therefore $\mathcal{T}$ is a topology on X. □

Definition 2. Let (X, d) be a metric space. Let $\mathcal{T}$ be the collection of all subsets of X that are open according to the metric d. Then $\mathcal{T}$ is said to be the topology on X **induced** by the metric d.

Example 4. Let X be an arbitrary set, and let d be the discrete metric on X. What is the topology induced by d, i.e., which subsets of X are open according to d? [4]

Definition 3. For any set X, the **discrete topology** on X is defined as $\mathcal{T} = \mathcal{P}(X)$, i.e., all subsets of X are declared to be open.

Definition 4. For any set X, the **indiscrete topology** on X is defined as $\mathcal{T} = \{\phi, X\}$.

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Q: Can you think of a metric d on a set X that induces the indiscrete topology on X?
Ans: If X has only one element, then this is possible. How? If X has two or more elements, then this is impossible (proved in HW).

According to Theorem 2 above, every metric space is a topological space. Is the converse also true, that is, is every topological space **metrizable** (i.e., does there exist a metric which induces the given topology)? We just saw that the answer is no (the indiscrete topology on a set with two or more elements). So it's natural to ask: which topological spaces are metrizable? The answer is a "big" theorem, which we will *not* get into:
Theorem: A topological space is metrizable iff $\cdots$.

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[1] No. Why? (Which of the three conditions isn't satisfied?)
[2] Yes. Why?
[3] No: $\bigcup_{n=1}^{\infty}(0,n) = (0,\infty) \notin \mathcal{T}$.
[4] $\mathcal{T} = \mathcal{P}(X)$ = the powerset of X = the set of all subsets of X (i.e., all subsets of X are open).
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Continuity

Definition 5. Let X and Y be two topological spaces. We say a function $f : X \to Y$ is **continuous** iff for every open set $U \subseteq Y$, its preimage $f^{-1}(U) \subseteq X$ is open in X.

Q: Let (X, d_X) and (Y, d_Y) be two metric spaces. So they are also topological spaces. T or F: A function $f : X \to Y$ is continuous according to the metric space definition iff it is continuous according to the topological definition? [5]

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Subspaces

Let X be a topological space, and A a subset of X. Then A "inherits" a topology from X in a natural way:

Definition 6. Let A be a subset of a set X. Given a topology $\mathcal{T}$ on X, we define the **subspace topology** (also called the *relative topology*) on A by $\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\}$. $(A, \mathcal{T}_A)$ is said to be a **subspace** of $(X, \mathcal{T})$. By the following theorem, $\mathcal{T}_A$ is a topology on A. We say $\mathcal{T}_A$ is **induced** on A by $\mathcal{T}$

Theorem 3. Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Then the subspace topology on A is a topology.

Proof: Homework.

Example 5. Let $X = \mathbb{R}$ with the standard topology (i.e., the topology induced by the Euclidean metric). Let $A = [2, 5) \subseteq X$. Let $\mathcal{T}_A$ be the topology that A inherits from X.

Q: Is the set $[2, 3)$ open, closed, or neither in A? Ans: It is open! Why? Because $[2, 3)$ is the intersection of A with some open set $U \subseteq X$. Can you find such a U? (There is more than one U that works here.)

To emphasize: $[2, 3)$ is open in A but is not open in X.

Q: Is the set $(2, 3)$ open in A? [6]

Q: Is the set $(2, 3]$ open in A? [7]

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Topological equivalence, and homeomorphisms

Example 6. Let $A = [0, 1] \subseteq \mathbb{R}$, $B = [0, 2] \subseteq \mathbb{R}$ (we give both A and B the subspace topology from $\mathbb{R}$). Intuitively, A and B are equivalent in some sense: stretching A to make it twice as long turns it into B.

There is a precise definition for what topological equivalence means.

Definition 7. Two topological spaces X and Y are **homeomorphic** (or *topologically equivalent*) iff $\exists f : X \to Y$ such that
1. f is 1-1 and onto;
2. f and f^{-1} are both continuous (or, equivalently, A is open in X iff $f(A)$ is open in Y).

When two spaces X and Y are homeomorphic, we write $X \simeq Y$. The function f is called a **homeomorphism** from X to Y.

Example 7. Continuing our last example... Can you prove $A \simeq B$, i.e., can you find a homeomorphism $f : A \to B$? [8]

Example 8. Let $A = [0, 2] \subseteq \mathbb{R}$, $B = [0, 1] \cup [2, 3] \subseteq \mathbb{R}$. Is A homeomorphic to B? [9]

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[5] True. Why?
[6] Yes. Why?
[7] No. Why?
[8] Define $f : A \to B$ by $f(x) = 2x$.
[9] No. We'll see a proof later.

It can be very difficult to show that two topological spaces are *not* homeomorphic: If you can't find a homeomorphism, maybe you're just not looking hard enough.

We will soon be able to prove $A \not\simeq B$ in the last example. Outline of the proof:
Step 1. Prove that A is connected (has only one piece), but B is not (it has two pieces).
Step 2. Prove that connectedness is a *topological invariant*, i.e., it's a property that is preserved under homeomorphisms; i.e., if A is connected and $A \simeq B$, then B is connected.