Definitions

Definition 1. A **metric space** M is a set X and a function $d : X \times X \to [0, \infty)$ such that $\forall x, y, z \in X$ 1. $d(x, y) = 0$ iff $x = y$; 2. $d(x, y) = d(y, x)$ (d is symmetric); 3. $d(x, z) \le d(x, y) + d(y, z)$ (triangle inequality).

Definition 2. Given a metric space M, a point $x \in M$, and a real number $r \ge 0$, the **ball** of radius r around x is defined as $B_r(x) = \{y \in M \mid d(x, y) < r\}$

Definition 3. A subset A of a metric space M is said to be **open** iff $\forall x \in A$, $\exists r > 0$ such that $B_r(x) \subset A$.

Definition 4. A subset A of a metric space M is said to be **closed** iff its complement $A^c = M - A$ is open.

Definition 5. The **closed ball** of radius r around x is defined as $\overline{B_r(x)} = \{y \in M \mid d(x, y) \le r\}$

Definition 6. Let A be a subset of a metric space M. A point $x \in M$ is said to be a **limit point** of A iff every ball around x contains a point of A other than x.

Definition 7. Given a subset A of a metric space M, its **interior** A° is defined as the set of all points $x \in A$ such that some open ball around x is a subset of A.

Definition 8. Given a subset A of a metric space M, its **closure** $\overline{A}$ is defined as A union the set of all limit points of A. The **boundary** of A is defined as $\partial A = \overline{A} - A^\circ$.

Definition 9. Let M_1, M_2 be metric spaces, with d_1 and d_2 as their corresponding distance functions. A function $f : M_1 \to M_2$ is said to be **continuous at** $a \in M_1$ iff $\forall \epsilon > 0$, $\exists \delta > 0$ such that $d_1(a, x) < \delta$ implies $d_2(f(a), f(x)) < \epsilon$. We say f is **continuous** if it is continuous at every point in M_1 .

Definition 10. A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X satisfying: 1. ϕ and X are in $\mathcal{T}$. 2. The union of any collection of sets in $\mathcal{T}$ is in $\mathcal{T}$. 3. The intersection of any finite collection of sets in $\mathcal{T}$ is in $\mathcal{T}$. The last two conditions are often stated as: $\mathcal{T}$ is **closed under** unions and finite intersections. The pair $(X, \mathcal{T})$ is called a **topological space**. The elements of $\mathcal{T}$ are called **open** sets. A subset $A \subset X$ is **closed** iff its complement in X is open.

Definition 11. Let (X, d) be a metric space. Let $\mathcal{T}$ be the collection of all subsets of X that are open according to the metric d. Then $\mathcal{T}$ is said to be the topology on X **induced** by the metric d.

Definition 12. For any set X, the **discrete topology** on X is defined as $\mathcal{T} = \mathcal{P}(X)$, i.e., all subsets of X are declared to be open.

Definition 13. For any set X, the **indiscrete topology** on X is defined as $\mathcal{T} = \{\phi, X\}$.

Definition 14. Let X and Y be two topological spaces. We say a function $f : X \to Y$ is **continuous** iff for every open set $U \subset Y$, its preimage $f^{-1}(U) \subset X$ is open in X.

Definition 15. Let A be a subset of a set X. Given a topology $\mathcal{T}$ on X, we define the **subspace topology** (also called the *relative topology*) on A by $\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\}$. $(A, \mathcal{T}_A)$ is said to be a **subspace** of $(X, \mathcal{T})$. $\mathcal{T}_A$ is said to be **induced** by $\mathcal{T}$.

Definition 16. Two topological spaces X and Y are **homeomorphic** (or *topologically equivalent*) iff $\exists f : X \to Y$ such that 1. f is 1-1 and onto; 2. f and f^{-1} are both continuous. When two spaces X and Y are homeomorphic, we write $X \simeq Y$. The function f is called a **homeomorphism** from X to Y.

Definition 17. A topological space X is **connected** iff it is *not* equal to the union of two disjoint nonempty open subsets.

Definition 18. Let X be a set, and $A \subset X$. A collection F of subsets of X is called a **cover** of A iff $A \subset \bigcup_{B \in F} B$. "Cover" is both a noun and a verb: F is a cover of A; F covers A.

Definition 19. Let X be a set, and $A \subset X$. Let F be a cover of A. A **subcover** of F is a set $F' \subset F$ that covers A.

Definition 20. Let X be a topological space, and F a cover of $A \subset X$. F is said to be an **open cover** of A if every element of F is open in X.

Definition 21. A topological space X is **compact** iff every open cover of X has a finite subcover.

Definition 22. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. Their **product** is defined by: A set U is open in $X \times Y$ iff it is a (finite or infinite) union of sets of the form $A \times B$, where A is open in X and B is open in Y. We write: $(X \times Y,\ \mathcal{T})$, where $\mathcal{T} = \{\bigcup A_\alpha \times B_\alpha \mid A_\alpha \in \mathcal{T}_X,\ B_\alpha \in \mathcal{T}_Y\}$.

Definition 23. Let X be a set, $\sim$ an equivalence relation on X, and Q the set of all equivalence classes of $\sim$, i.e, $Q = \{[x] \mid x \in X\}$. Define a map $q : X \to Q$ by: For each $x \in X$, $q(x) = [x]$. The map q is called the **quotient map** (or the *identification map*) from X to Q. We sometimes write $X/\sim$ instead of Q.

Definition 24. Let $(X, \mathcal{T}_X)$ be a topological space, $\sim$ an equivalence relation on X, and $q : X \to Q$ the corresponding quotient map. The **quotient topology** on Q is defined as $\mathcal{T}_Q = \{U \subset Q \mid q^{-1}(U) \in \mathcal{T}_X\}$. In other words, U is declared to be open in Q iff its preimage $q^{-1}(U)$ is open in X. The pair $(Q, \mathcal{T}_Q)$ is called the **quotient space** (or the *identification space*) obtained from $(X, \mathcal{T}_X)$ and the equivalence relation $\sim$.

Definition 25. Given a point x in a topological space X, a **neighborhood** of x is any open set that contains x.

Definition 26. A topological space X is said to be **locally homeomorphic** to a topological space Y iff every point in X has some neighborhood that is homeomorphic to Y.

Definition 27. The **n-dimensional upper half-space** is defined as $\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n \ge 0\}$ When $n = 2$, $\mathbb{R}^2_+$ is also called the **upper half-plane**.

Definition 28. A topological space X is called an n-**dimensional manifold** (n-mfd for short) if it is Hausdorff, Second Countable, and every point $x \in X$ has a neighborhood that is homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+$. A point that has a neighborhood homeomorphic to $\mathbb{R}^n_+$ but not to $\mathbb{R}^n$ is called a **boundary point**. The set of all such points (if any) is called the **boundary** of X, denoted by ∂X. If $\partial X \neq \phi$, then, for emphasis, X is sometimes called a **manifold with boundary**.

Definition 29. A topological space X is said to be **Hausdorff** iff every two points in X have disjoint neighborhoods.

Definition 30. A manifold is said to be **closed** if it is compact and has no boundary.

Definition 31. Let X and Y be topological spaces. An **embedding** of X into Y is a map $f : X \to Y$ such that f is a homeomorphism from X onto its image $f(X)$, where $f(X)$ is given the subspace topology as a subset of Y.

Definition 32. Let M and N be connected n-manifolds. Let $x \in M$, $y \in N$. Let B_x and B_y be neighborhoods of x and y, respectively, such that they are homeomorphic to open balls in $\mathbb{R}^n$, and their boundaries ∂B_x and ∂B_y are homeomorphic to S^{n-1}. The quotient space obtained by gluing $M - B_x$ to $N - B_y$ along their sphere boundaries is called the **connected sum** of M and N, and is denoted by $M\#N$.

Definition 33. Let (X, d) be a metric space, and let $A \subset X$. Given $\epsilon > 0$, the ϵ**-neighborhood** of A in X is defined as the set of all points in X whose distance is less than ϵ from some point in A: $N_\epsilon(A) = \{x \in X \mid (\exists a \in A) d(x, a) < \epsilon\}$.

Definition 34. $M = [0, 1]^2/\{(0, y) \sim (1, 1 - y)\}$ is called the **Möbius band** (or Möbius strip). $K = [0, 1]^2/\{(x, 0) \sim (x, 1)\ ,\ (0, y) \sim (1, 1 - y)\}$ is called the **Klein bottle**. $P = [0, 1]^2/\{(x, 0) \sim (1 - x, 1)\ ,\ (0, y) \sim (1, 1 - y)\}$ is called the **projective plane**, more commonly denoted by $\mathbb{R}\mathrm{P}^2$.

Definition 35. For $n \geq 1$, the n**-hole torus** is the connected sum of n tori, denoted by nT^2. Similarly, the connected sum of n projective planes is denoted by $n\mathbb{R}\mathrm{P}^2$.

Definition 36. Let A be a subset of a connected topological space X. To say A **separates** X means $X - A$ is not connected.

Definition 37. Let M be a 2-manifold, and let D^2 denote the closed unit disk in $\mathbb{R}^2$. Given an embedding $h : D^2 \to M$, the **mirror-image** $h' : D^2 \to M$ of h is defined by $h'(x, y) = h(-x, y)$. We say M is **non-orientable** if there is an embedding $h : D^2 \to M$ that is isotopic to its mirror-image. Otherwise, we say M is **orientable**.

Definition 38. For each $n \geq 1$, the surface nT^2 is said to have **genus** n. S^2 is said to have **genus** 0. ($n\mathbb{R}\mathrm{P}^2$ is said to have **genus** $n/2$.)

Definition 39. For $n \geq 0$, the n**-sphere** is defined as: $S^n = \{\vec{x} \in \mathbb{R}^{n+1} \mid d(\vec{x}, \vec{0}) = 1\}$.

Definition 40. Let X be a topological space. An embedded circle in X is called a **simple closed curve** (scc). (*Simple* means not self-intersecting; *closed* means it's a loop – no endpoints.)

Definition 41. Let X be a topological space, with $A \subset B \subset X$. We say A **bounds** B iff $A = \partial B$.

Definition 42. Let X and Y be topological spaces. We say an embedding $f : X \to Y$ is **isotopic** to another embedding $g : X \to Y$, denoted $f \approx g$, iff there exists a continuous map $H : X \times I \to Y$ such that $\forall x \in X$ 1. $H(x, 0) = f(x)$. 2. $H(x, 1) = g(x)$. 3. $\forall t \in I$, $H(\cdot, t)$ is an embedding of X into Y. We say H is an **isotopy** from f to g.

Definition 43. Given a fixed $t \in I$, the map $H(\cdot, t) : X \to Y$ is defined by: $\forall x \in X$, $x \mapsto H(x, t)$.
(Read the symbol "$\mapsto$" as "maps to", or "is sent to".)
Instead of $H(\cdot, t) : X \to Y$ we often write $H_t : X \to Y$. They are equivalent.

Definition 44. An embedded circle in $\mathbb{R}^3$ is called a **knot**. A set of (one or more) disjoint embedded circles in $\mathbb{R}^3$ is called a **link**.

Definition 45. Let X be a topological space. A **path** (or a curve) in X is a continuous map $p : I \to X$. A path whose **initial point** $p(0)$ equals its **terminal point** $p(1)$ is called a **loop** (or a closed curve).

Definition 46. Let X and Y be topological spaces, and f and g continuous maps from X to Y. A **homotopy** from f to g is a continuous map $H : X \times I \to Y$ such that 1. $H_0 = f$. 2. $H_1 = g$. We say f is **homotopic** to g, and write $f \sim g$.

Definition 47. Let X be a topological space. A loop whose image is just one point in X is called a **trivial loop**. A loop is said to be **null-homotopic** if it is homotopic as a loop to a trivial loop in X.

Theorems

Theorem 1. A subset A of a metric space M is closed iff it contains all its limit points.

Theorem 2. Given metric spaces M and N, a function $f : M \to N$ is continuous iff the preimage (or inverse image) of every open set is open; i.e., for every open set $U \subset N$, $f^{-1}(U)$ is an open subset of M.

Theorem 3. Every metric space is a topological space. More precisely, given a metric (i.e., a distance function) d on a set X, let $\mathcal{T}$ be the collection of subsets of X that are open according to d. Then $\mathcal{T}$ satisfies the definition of being a topology. $\mathcal{T}$ is said to be the topology **induced** by the metric d.

Theorem 4. Let $(X, \mathcal{T})$ be a topological space, and let $A \subset X$. Then the subspace topology on A is a topology.

Theorem 5. Let $f : X \to Y$ be a continuous map between two topological spaces. Then for every subspace $A \subset X$, the restriction of f to A, i.e., $f|_A : A \to Y$, is continuous.

Theorem 6. If $A \simeq B$, then A is connected iff B is connected.

Theorem 7. $\mathbb{R}$ is connected.

Theorem 8. $A \subset \mathbb{R}$ is connected iff A is an interval (open, closed, or half open).

Theorem 9. The continuous image of a connected set is connected; i.e, if $f : X \to Y$ is a continuous map between topological spaces, and if X is connected, then f(X) is connected.

Theorem 10. If $A \simeq B$, then A is connected iff B is connected.

Theorem 11. A topological space X is connected iff it contains no proper subset which is both open and closed in X.

Theorem 12. If A and B are connected subspaces of a topological space X, and if $A \cap B \neq \phi$, then $A \cup B$ is connected.

Theorem 13. 1. $\mathbb{R}$ is not compact. 2. (Heine-Borel) Every closed interval $[a, b] \subset \mathbb{R}$ is compact.

Theorem 14. The continuous image of a compact set is compact; i.e., if $f : X \to Y$ is a continuous map between two topological spaces, and if X is compact, then $f(X)$ is compact.

Theorem 15. (Classification of 1-manifolds) Every 1-manifold is homeomorphic to $[0, 1]$ or $(0, 1)$ or $[0, 1)$ or S^1.

Theorem 16. For $m \leq n$, S^n cannot be embedded in $\mathbb{R}^m$.

Theorem 17. Every closed 2-manifold that can be embedded in $\mathbb{R}^3$ is homeomorphic to S^2 or to an n-hole torus (= the connected sum of n tori) for some $n \geq 1$.

Corollary 18. Every closed 2-manifold is homeomorphic to either S^2 or nT^2 or $n\mathbb{R}\mathrm{P}^2$, for some $n \geq 1$.

Theorem 19. $\mathbb{R}\mathrm{P}^2$ cannot be embedded in $\mathbb{R}^3$.

Theorem 20. (1) The boundary of a Möbius band is a circle: $\partial M \simeq S^1$. (2) Gluing a Möbius band and a closed disk along their circle-boundaries yields a projective plane: $M \cup_\partial \overline{D^2} \simeq \mathbb{R}\mathrm{P}^2$. (3) Gluing two Möbius bands along their circle-boundaries yields a Klein bottle: $M \cup_\partial M \simeq K$.

Theorem 21. Every closed 2-manifold that cannot be embedded in $\mathbb{R}^3$ is homeomorphic to the connected sum of n projective planes for some $n \geq 1$.

Theorem 22. $T^2 \# \mathbb{R}\mathrm{P}^2 \simeq 3\mathbb{R}\mathrm{P}^2$.

Theorem 23. A torus is not homeomorphic to a Klein bottle.

Theorem 24. $S^2 \not\simeq \mathbb{R}\mathrm{P}^2$.

Theorem 25. Every compact surface (with boundary) is homeomorphic to some closed surface minus a finite set of open disks. In other words, every compact surface can be obtained by removing finitely many open disks from one of S^2, nT^2, or $n\mathbb{R}\mathrm{P}^2$.

Theorem 26. For all $n \geq 1$, S^2 and nT^2 are orientable, while $n\mathbb{R}\mathrm{P}^2$ is non-orientable.

Theorem 27. For $n \geq m$, S^n cannot be embedded in $\mathbb{R}^m$.

Theorem 28. The only connected 3-manifold in which an embedded 2-sphere bounds a ball on both sides is the 3-sphere.

Theorem 29. $S^3 \not\simeq S^2 \times S^1$.

Theorem 30. $\approx$ is an equivalence relation.

Defs & Thms from HW

Restriction of Continuous Maps Lemma: Let $f : X \to Y$ be a continuous map between two topological spaces. Then for every subspace $A \subset X$, the restriction of f to A, i.e., $f|_A : A \to Y$, is continuous.

Definition 48. A subset A of a metric space X is **bounded** iff for some positive real number r and for some point $x \in X$, $A \subset B_r(x)$.

Definition 49. We say a function $f : X \to Y$, where X and Y are metric spaces, is **bounded** iff its image $f(X)$ is a bounded subset of Y.

Jordan Curve Theorem: Every embedded circle $C \subset \mathbb{R}^2$ separates $\mathbb{R}^2$.

Definition 50. Two simple loops $f_0 : I \to X$ and $f_1 : I \to X$ are **isotopic as simple loops** in X if there is a continuous map $H : I \times I \to X$ such that $H_0 = f_0$, $H_1 = f_1$, and for all $t \in I$, $H_t : I \to X$ is a simple loop.

Definition 51. Let X be a topological space. A loop whose image is just one point in X is called a **trivial loop**. A loop is said to be **null-homotopic** iff it is homotopic as a loop to a trivial loop in X.

Theorem 31. (Brouwer Fixed Point Theorem, Dimension 1) Let $f : I \to I$ be a continuous map. Then f has a fixed point.

Theorem 32. (Borsuk-Ulam Theorem, Dimension 2) Let $f : S^2 \to \mathbb{R}^2$ be continuous. Then there exist antipodal points $p, -p \in S^2$ such that $f(p) = f(-p)$.

Theorem 33. (Brouwer Fixed Point Theorem, Dimension n) Every continuous map $f : \overline{B^n} \to \overline{B^n}$ has a fixed point.
