

The *long line* is an example of a connected, Hausdorff topological space that's locally homeomorphic to  $\mathbb{R}$ , but is not a manifold, because it is not second countable. Roughly speaking, it is  $S_\Omega \times [0, 1)$ , where  $S_\Omega$  is the smallest uncountable ordinal. To describe and understand it in detail, we first need some definitions. I am borrowing most of the following from Munkres's book, *Topology*, Second Edition, Prentice Hall.

*Definition* A relation  $R$  on a set  $X$  is an **order relation** if  $\forall x, y \in X$ , (1)  $xRy$  or  $yRx$ , (2)  $\neg(xRx)$ , and (3)  $R$  is transitive.

*Definition* A set  $X$  is **well ordered** by an order relation  $<$  if every nonempty subset of  $X$  has a "smallest" element.

We construct the long line as follows. Let  $S_\Omega$  be the smallest uncountable ordinal (if you don't know what that is, just think of it as an a well-ordered uncountable set). Let  $L = S_\Omega \times [0, 1) - \{(s_0, 0)\}$ , where  $s_0$  is the smallest element of  $S_\Omega$ . Order  $L$  with the dictionary ordering, i.e.,  $(s, t) < (s', t')$  iff  $s < s'$  or  $[s = s'$  and  $t < t']$ .

We give  $L$  the **order topology**, which means a subset of  $L$  is open iff it is a union of open intervals  $(a, b)$  with  $a, b \in L$ ,  $a < b$ . Then  $L$  with this topology is called the **long line**.



Extra Credit Problems

1. Explain why we choose  $L = S_\Omega \times [0, 1)$  minus its smallest element instead of just letting  $L = S_\Omega \times (0, 1]$ . Don't read the following hint until you've thought hard. Hint: The second way would not produce a connected space; why?

*Definition* Let  $\mathcal{T}$  be a topology on a set  $X$ . A **basis** for  $\mathcal{T}$  is a subset  $\beta \subseteq \mathcal{T}$  such that  $\forall U \in \mathcal{T}$  and  $\forall x \in U$ ,  $\exists B \in \beta$  with  $x \in B \subseteq U$ .

2. (a) Prove that the long line is not **second countable**, i.e, it does not have a countable basis.  
 (b) Prove that the long line cannot be embedded in  $\mathbb{R}$ , or in any  $\mathbb{R}^n$ . Hint:  $\mathbb{R}^n$  has a countable basis.