

*Definition* Let  $X$  be a topological space. A loop whose image is just one point in  $X$  is called a **trivial loop**. A loop is said to be **null-homotopic** iff it is loop-homotopic to a trivial loop in  $X$ . (For a definition of *loop-homotopic* see the previous set of homework problems.)

1. (a) Show that every loop in  $\mathbb{R}^2$  is null-homotopic.  
 (b) Show that any two loops in  $\mathbb{R}^2$  are null-homotopic to each other. Do this twice, using each method below once.  
 Method 1: Use Part (a) above, and the fact that being loop-homotopic is an equivalence relation. (We've shown that "being homotopic" and "being isotopic" are equivalence relations. We haven't shown that "being loop-homotopic" is an equivalence relation, but you may use this fact; its proof is very similar to that of the former two.)  
 Method 2: Construct a loop-homotopy. (Don't forget to show that in every "frame" of your loop-homotopy you in fact see a loop.)
2. (a) Draw three loops on the torus,  $T^2$ , such that no two of them are loop-homotopic to each other. (No proof necessary). Can you find more than three?  
 (b) How many loops are there on the annulus  $S^1 \times I$  such that no two of them are loop-homotopic to each other? Support your answer by constructing the loops (but you don't need to prove that no two of them are loop-homotopic to each other).
3. Recall that  $\mathbb{RP}^2$  is defined as (this is one of two definitions we have seen): the closed unit disk with **antipodal** (i.e., opposite) points on its boundary identified;  $\mathbb{RP}^2 = D^2 / \{\forall x \in \partial D^2, x \sim -x\}$ . Let  $q : D^2 \rightarrow \mathbb{RP}^2$  be the quotient map.
  - (a) Let  $A$  be the horizontal diagonal in  $D^2$ , i.e.,  $A = \{(x, y) \in D^2 \mid y = 0\}$ . Let  $\alpha = q(A) \subset \mathbb{RP}^2$ . Then  $\alpha$  is a closed curve in  $\mathbb{RP}^2$ . Why? Technically,  $\alpha$  is not really a loop. Why?
  - (b) Give a homeomorphism  $h$  from  $I$  to  $A$ .
  - (c) Explain why the composition  $q|_A \circ h$  is a loop in  $\mathbb{RP}^2$ . What is the image of this loop? Do you think this loop is null-homotopic (just Y or N, without proof)?
  - (d) Define  $g : I \rightarrow D^2$  by  $g(s) = (\cos(2\pi s), \sin(2\pi s))$ . Then  $q|_{g(I)} \circ g$  is a loop in  $\mathbb{RP}^2$ . Why? Prove that the loop  $q|_{g(I)} \circ g$  is null-homotopic.
4. How many loops are there on  $\mathbb{RP}^2$  such that no two of them are loop-homotopic to each other? (Proof not necessary.)

#### Extra Credit Problems

5. Prove that the Möbius band  $M$  is not orientable, by giving an embedding  $h : D^2 \rightarrow M$  and an isotopy between  $h$  and its mirror image.
6. (a) Represent  $3T^2$  as a polygon with edges identified appropriately.  
 (b) Represent  $3\mathbb{RP}^2$  as a polygon with edges identified appropriately.