

1. (a) Given that S^n is defined as the boundary of the closed unit ball in \mathbb{R}^{n+1} , describe S^0 .
 - (b) The intersection of S^2 with the xy -plane in \mathbb{R}^3 is S^1 , a circle of radius 1. We call this a *great circle*, since no other circle on S^2 has a larger radius. Similarly, the intersection of S^2 with any plane through the origin in \mathbb{R}^3 is called a **great circle**. What is the intersection of two great circles?
 - (c) Now one dimension higher. Denote points in \mathbb{R}^4 by (x, y, z, w) . Prove *rigorously* that the intersection of S^3 with the xyz -hyperplane in \mathbb{R}^4 is S^2 . We call such a 2-sphere a **great sphere**.
 - (d) What is the intersection of two great spheres? Prove your answer rigorously for the intersection of the great sphere cut out by the xyz -hyperplane with the great sphere cut out by the yzw -hyperplane.
 - (e) The general form equation of a plane in \mathbb{R}^3 is: $ax + by + cz = d$, where a, b, c, d are constants. The general form equation of an $n - 1$ -dimensional hyperplane in \mathbb{R}^n is: $a_1x_1 + a_nx_n = b$, where a_i and b are constants. Think this way: in \mathbb{R}^n , you have n “degrees of freedom”. An equation gives one constraint, reducing the number of degrees of freedom to $n - 1$; hence the solutions to the equation form an $(n - 1)$ -dimensional manifold (which is a hyperplane if the equation is linear). Now, here’s the question: What is the intersection of two $(n - 1)$ -dimensional hyperplanes that pass through the origin in \mathbb{R}^n ? Explain your reasoning.
 - (f) Give a definition for a great $(n - 1)$ -sphere in $S^n \subset \mathbb{R}^{n+1}$. Describe the intersection of two great $(n - 1)$ -spheres in S^n , and explain your reasoning (it doesn’t have to be a rigorous proof, but only a clear and convincing explanation; though a rigorous proof wouldn’t be bad either!). Hint: Use the previous part of this problem.
2. A torus T^2 can be defined as a “solid square” $I^2 = [0, 1] \times [0, 1]$ with its opposite edges identified (with the “appropriate” orientations): $T^2 = I^2/R$, where R denotes the equivalence relation $(x, 0) \sim (x, 1)$, $(0, y) \sim (1, y)$. Similarly, a 3-dimensional torus T^3 can be defined as a solid cube I^3 with its opposite faces identified (with the “appropriate” orientations). Make this precise by giving an appropriate definition for R' : $T^3 = I^3/R'$, where R' is \dots .
 3. T^2 can also be defined as $S^1 \times S^1$. We can informally explain how this definition is equivalent to the above definition ($T^2 = I^2/R$) by arguing as follows. For every $t \in I$, the two endpoints of $I \times \{t\} \subset I^2$ are identified into one point; so each $I \times \{t\}$ becomes homeomorphic to $S^1 \times \{t\}$. Therefore, I^2/R is homeomorphic to $S^1 \times I$ with $S^1 \times \{0\}$ identified with $S^1 \times \{1\}$ (with the “right” orientation). Thus, we get $S^1 \times S^1$. Give a similar informal argument to show $I^3/R' \simeq S^1 \times S^1 \times S^1$.
 4. (a) What familiar space is a punctured 3-sphere (S^3 minus one point) homeomorphic to? Briefly explain why.
 - (b) Let p be an arbitrary point in S^2 . Then, in $S^2 \times S^1$, $\{p\} \times S^1$ is a simple closed curve. Draw a schematic diagram of this. Call this simple closed curve C . Does C bound a disk in $S^2 \times S^1$?
 - (c) What familiar space is $(S^2 \times S^1) - (N_\epsilon(C))^\circ$ (i.e., $(S^2 \times S^1)$ minus the interior of an ϵ -neighborhood of C) homeomorphic to?
 5. (a) It is possible to travel in \mathbb{R}^3 from the point $(-1, 0, 0)$ to the point $(1, 0, 0)$ by walking along straight line segments and without ever touching the y -axis. Explain how. How about without ever the yz -plane?
 - (b) It is possible to travel in \mathbb{R}^4 from the point $(-1, 0, 0, 0)$ to the point $(1, 0, 0, 0)$ by walking along straight line segments and without ever touching the yz -plane $\{(x, y, z, w) \in \mathbb{R}^4 \mid x = w = 0\}$. Explain how.

6. A **2-component link** consists of two disjoint circles embedded in \mathbb{R}^3 . For example, let $X \subset \mathbb{R}^3$ be the unit circle in the xy -plane centered at the origin, and let $Y \subset \mathbb{R}^3$ be the unit circle in the yz -plane centered at $(0, 1, 0)$. Then $X \cup Y$ is a 2-component link; in fact, it has its own name: the **Hopf link** (named after a mathematician). Draw a picture of this link, showing the three axes in \mathbb{R}^3 and the coordinates of the circles' centers. The two components of the link, X and Y , cannot be “pulled apart”; more precisely, if we let Y' be a unit circle centered at $(0, 5, 0)$, then $X \cup Y$ is not isotopic to $X \cup Y'$. This is not easy to prove rigorously with what we know so far, but should be clear intuitively—do you see it?

Now, \mathbb{R}^3 can be viewed as a subset of \mathbb{R}^4 : $\mathbb{R}^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid w = 0\}$. Then $X \cup Y \subset \mathbb{R}^3 \subset \mathbb{R}^4$. Explain informally how $X \cup Y$ can be “pulled apart” in \mathbb{R}^4 .

7. Give a detailed and rigorous proof that if $n > m$, then S^n cannot be embedded in \mathbb{R}^m . (Use the theorem that says S^n cannot be embedded in \mathbb{R}^n .)

Extra Credit Problems

8. (a) Explain how $S^2 \times S^1$ can be viewed as two solid tori glued together along their boundaries.
(b) Explain how S^3 can be viewed as two solid tori glued together along their boundaries. Hint: view it as the boundary of $B^2 \times B^2$ ($\simeq B^4$).
(c) Explain why the above does *not* imply $S^3 \simeq S^2 \times S^1$.
9. Give an embedding of $\mathbb{R}P^2$ into \mathbb{R}^4 .