

1. Let $A = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x, y, z \leq 1 \text{ and at least two of } x, y, z \text{ are in the set } \{0, 1\}\}$. Let $F = \partial(N_{0.1}(X))$. Draw a picture of A and a picture of F . Show, using pictures, that $F \simeq nT^2$ for some n .
2. Let (X, d) be a metric space, and let $A \subset X$. True or False: $\forall \epsilon > 0, N_\epsilon(A) = \bigcup_{a \in A} B_\epsilon(a)$. Prove your answer.
3. Take a strip of paper and glue its two shorter edges in such a way as to create a Möbius band. In the following, let $M = [0, 1]^2 / \{(0, y) \sim (1, 1 - y)\}$ denote the Möbius band, and let $q : [0, 1]^2 \rightarrow M$ be the corresponding quotient map. Let $A = [0, 1] \times \{1/2\} \subset [0, 1]^2$.
 - (a) Explain why $q(A) \subset M$ is a circle. Draw this circle on the Möbius band that you created. We call this circle the **core** of the Möbius band.
 - (b) Use a pair of scissors to cut your Möbius band along its core. Is this a separating circle? Use schematic diagrams to prove your answer.
 - (c) Let $B = [0, 1] \times \{1/4\} \cup [0, 1] \times \{3/4\}$. Is $q(B)$ a circle? Use diagrams to explain.
 - (d) Create a new Möbius band, and draw $q(B)$ on it. *Before* cutting along it, draw diagrams to try to predict whether or not $M - q(B)$ is connected. Now cut. Was your prediction correct? Draw diagrams to explain why $q(B)$ does or does not separate M .
4. Let $f : X \rightarrow Y$ be a homeomorphism between topological spaces. Let $A \subset X$. True or false: A separates X iff $f(A)$ separates Y . Prove your answer.
5. Prove, using one of the theorems of Section 9, that the torus and the Klein Bottle are not homeomorphic.
6. Prove, without using Theorem 2 of Section 9, that $\mathbb{R}P^2 \not\cong S^2$. Hint: (i) *Jordan Curve Theorem*: Every embedded circle $C \subset S^2$ separates S^2 (you may use this without proving it). (ii) Show there exists a non-separating circle $C \subset \mathbb{R}P^2$ (from above we know that the Möbius band has a non-separating circle). (iii) Assume $\mathbb{R}P^2 \simeq S^2$, and use (i) and (ii) to get a contradiction.
7. (a) Prove that gluing two Möbius bands along their circle boundaries yields a Klein bottle: $M \cup_\partial M \simeq K$. (You may find it easier to prove that a Klein bottle can be cut up into two Möbius bands.)
 - (b) Use Part (a) above to prove that the Klein bottle is homeomorphic to the connected sum of two projective planes.
 - (c) Use the Extra Credit Problem below to prove that $T^2 \# \mathbb{R}P^2 \simeq K \# \mathbb{R}P^2$.
8. Use the fact that \mathbb{R} is connected to prove that S^1 is connected.

Extra Credit Problems

9. Prove that $T^2 \# \mathbb{R}P^2 \simeq 3\mathbb{R}P^2$.
10. Embed the Möbius band (by drawing a picture of it) in \mathbb{R}^3 in such a way that its boundary is the unit circle in the xy -plane.