
3. True or false: If A and B are non-disjoint connected subspaces of a topological space X , then $A \cup B$ is connected. Prove your answer.

Solution:

True. Outline of Proof: Let $C = A \cup B$. Suppose toward contradiction that C is not connected. Then C is the union of two disjoint nonempty open subsets, U and V .

Case 1. $A \not\subset U$ and $A \not\subset V$. Then $A \cap U$ and $A \cap V$ are disjoint nonempty open subsets of A whose union is A ; but this contradicts the hypothesis that A is connected.

Case 2. $A \subset U$ or $A \subset V$. WLOG, assume $A \subset U$. Then $B \cap V \neq \emptyset$, otherwise V would be empty. Since B intersects A , $B \cap U \neq \emptyset$. Now, as in Case 1, B is not connected, which is a contradiction.
