Review definitions of neighborhood, locally homeomorphic, and manifold.

Recall that, in the definition of manifold, we can replace "locally homeomorphic to an open ball in $\mathbb{R}^n$" with "locally homeomorphic to $\mathbb{R}^n$."

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Example 1. Is the open rectangle $(0,1) \times (0,2) \subset \mathbb{R}^2$ a manifold? Yes. Of what dimension? 2.

Is the closed rectangle $[0,1] \times [0,2] \subset \mathbb{R}^2$ a manifold? No. Why?

We'd like to say that the closed rectangle is a manifold *with boundary.* Before defining this, we need another definition.

Definition 1. The n-**dimensional upper half-space** is defined as

$$\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$$

When $n = 2$, $\mathbb{R}^2_+$ is also called the **upper half-plane**.

Example 2. Draw a picture of what each of $\mathbb{R}^1_+$ and $\mathbb{R}^2_+$ looks like.

Example 3. Let $X = B_1(0,0) \cap \mathbb{R}^2_+$. Is X homeomorphic to $\mathbb{R}^2_+$? Does every point in X have a neighborhood that's homeomorphic to either $\mathbb{R}^2$ or $\mathbb{R}^2_+$? [1]

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Definition 2. (Overrides previous definition of manifold) A topological space X is called an n-**dimensional manifold** (n-mfd for short) if it is Hausdorff, Second Countable, and every point $x \in X$ has a neighborhood that is homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+$. A point that has a neighborhood homeomorphic to $\mathbb{R}^n_+$ but has no neighborhood that's homeomorphic to $\mathbb{R}^n$ is called a **boundary point**. The set of all such points (if any) is called the **boundary** of X, denoted by ∂X. If $\partial X \neq \phi$, then, for emphasis, X is sometimes called a **manifold with boundary**.

Remark. Depending on the context, the term *boundary* can have two different meanings: when applied to a subset A of a topological space, it means $\overline{A} - A^\circ$; but when applied to a manifold, it is defined according to the above definition. For a given topological space that's also a mfd, these two different types of boundary may happen to be the same set of points, but most often they are not! (Thus, the symbol ∂ has at least three different meanings in mathematics: two types of boundary, plus partial derivative.)

Example 4. Let $X = ([0,1] \times [0,1])/\{(0,y) \sim (1,y)\}$. Draw a picture of X. Is X a manifold? Yes. Is it a manifold with boundary? Yes. What is ∂X? [2]

Example 5. Let $X = [0,1] \times [0,1]/\{(0,y) \sim (\frac{1}{2}, y)\}$. Draw a picture of X. Is X a manifold? [3]

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Theorem 1. (Classification of 1-manifolds) Every connected 1-manifold is homeomorphic to $[0,1]$ or $(0,1)$ or $[0,1)$ or S^1.

Idea of Proof: What things can you create by joining or overlapping line segments end-to-end? Only these four.

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We are often interested in studying manifolds that are compact and have no boundary. (Why? One reason is that non-compact manifolds are usually more difficult to study, or at least different very from compact ones.) There is a name for such manifolds:

[1] Yes to both; why?

[2] X is homeomorphic to a cylinder, and it's boundary is homeomorphic to two disjoint circles: $([0,1] \times \{0\}/\{(0,0) \sim (1,0)\}\} \cup ([0,1] \times \{1\}/\{(0,1) \sim (1,1)\}\}$.

[3] No, why?

Definition 3. A manifold is said to be **closed** if it is compact and has no boundary.

Remark. Do not confuse the two (very) different meanings of *closed*; they depend on the context: A subset A of a topological space X is closed if $X - A$ is open in X (i.e., $X - A$ is in $\mathcal{T}$). A manifold is closed if it's compact and has no boundary.

Example 6. Which of the four connected 1-mfds are closed? Why is each of the others not closed? [4]

Hausdorff Spaces

Definition: A topological space X is said to be **Hausdorff** iff every pair of distinct points $x_1, x_2 \in X$ can be **separated** by open sets, i.e., there exist disjoint open sets $U_1, U_2 \subseteq X$ such that $x_i \in U_i$.

Example 7. Determine whether each of the following is Hausdorff.

(a) $\mathbb{R}^2$ with the standard topology (induced by the Euclidean metric).

(b) $\mathbb{R}^2$ with the discrete topology.

(c) $\mathbb{R}^2$ with the indiscrete topology. [5]

Example 8. Which of the following are manifolds? Why?

(a) $([0,2] \cup [5,7])/\{\forall x \in [0,1],\ x \sim (x+5)\}$. [6]

(b) $([0,2] \cup [5,7])/\{\forall x \in [0,1),\ x \sim (x+5)\}$.

Answer to (b): Every point does have a neighborhood that's homeomorphic to $\mathbb{R}$; nevertheless, this is not a manifold! Why? [7]

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[4] Only S^1 is closed. $[0,1]$ has boundary. $(0,1)$ isn't compact. $[0,1)$ has boundary and isn't compact.

[5] Yes, yes, no. Why?

[6] Not a mfd, since the point $[1] = [6] = \{1, 6\}$ in the quotient space does not have a neighborhood that's homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+$ for any n.

[7] Because it's not Hausdorff: the points 1 and 6 cannot be separated by open sets.