3. True or false: If A and B are non-disjoint connected subspaces of a topological space X, then $A \cup B$ is connected. Prove your answer.

Solution:

True. Outline of Proof: Let $C = A \cup B$. Suppose toward contradiction that C is not connected. Then C is the union of two disjoint nonempty open subsets, U and V.

Case 1. Both U and V intersect A. Then $A \cap U$ and $A \cap V$ are disjoint nonempty open subsets of A whose union is A; but this contradicts the hypothesis that A is connected.

Case 2. A is disjoint from U or V. WLOG, assume $A \cap V = \emptyset$. Then $B \cap V \neq \emptyset$, otherwise V would be empty. Now, since $A \cap V = \emptyset$, $A \subset U$. Therefore $B \cap U \neq \emptyset$ (since B intersects A). So, as in Case 1, B is not connected, which is a contradiction.