1. (a) Given that S^n is defined as the boundary of the closed unit ball in $\mathbb{R}^{n+1}$, describe S^0.

   (b) The intersection of S^2 with the xy-plane in $\mathbb{R}^3$ is S^1, a circle of radius 1. We call this a *great circle*, since no other circle on S^2 has a larger radius. Similarly, the intersection of S^2 with any plane through the origin in $\mathbb{R}^3$ is called a **great circle**. What is the intersection of two great circles?

   (c) Now one dimension higher. Denote points in $\mathbb{R}^4$ by (x, y, z, w). Prove *rigorously* that the intersection of S^3 with the xyz-hyperplane in $\mathbb{R}^4$ is S^2. We call such a 2-sphere a **great sphere**.

   (d) What is the intersection of two great spheres? Prove your answer rigorously for the intersection of the great sphere cut out by the xyz-hyperplane with the great sphere cut out by the yzw-hyperplane.

   (e) The general form equation of a plane in $\mathbb{R}^3$ is: $ax + by + cz = d$, where a, b, c, d are constants. The general form equation of an $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$ is: $a_1x_1+\cdots+a_nx_n = b$, where a_i and b are constants. Think this way: in $\mathbb{R}^n$, you have n "degrees of freedom". An equation gives one constraint, reducing the number of degrees of freedom to $n - 1$; hence the solutions to the equation form an $(n - 1)$-dimensional manifold (which is a hyperplane if the equation is linear). Now, here's the question: What is the intersection of two distinct $(n-1)$-dimensional hyperplanes that pass through the origin in $\mathbb{R}^n$? Explain your reasoning.

   (f) Give a definition for a great $(n - 1)$-sphere in $S^n \subset \mathbb{R}^{n+1}$. Describe the intersection of two distinct great $(n - 1)$-spheres in S^n, and explain your reasoning (it doesn't have to be a rigorous proof, but only a clear and convincing explanation; though a rigorous proof wouldn't be bad either!). Hint: Use the previous part of this problem.

2. A torus T^2 can be defined as a "solid square" $I^2 = [0, 1] \times [0, 1]$ with its opposite edges identified (with the "appropriate" orientations): $T^2 = I^2/R$, where R denotes the equivalence relation $(x, 0) \sim (x, 1)$, $(0, y) \sim (1, y)$. Similarly, a 3-dimensional torus T^3 can be defined as a solid cube I^3 with its opposite faces identified (with the "appropriate" orientations). Make this precise by giving an appropriate definition for R': $T^3 = I^3/R'$, where R' is $\cdots$.

3. T^2 can also be defined as $S^1 \times S^1$. We can informally explain how this definition is equivalent to the above definition ($T^2 = I^2/R$) by arguing as follows. For every $t \in I$, the two endpoints of $I \times \{t\} \subset I^2$ are identified into one point; so each $I \times \{t\}$ turns into $S^1 \times \{t\}$. Therefore, I^2/R is homeomorphic to $S^1 \times I$ with $S^1 \times \{0\}$ identified with $S^1 \times \{1\}$ (with the "right" orientation). Thus, we get $S^1 \times S^1$. Give a similar informal argument to show $I^3/R' \simeq S^1 \times S^1 \times S^1$.

4. (a) What familiar space is a punctured 3-sphere (S^3 minus one point) homeomorphic to? Briefly explain why.

   (b) Let p be an arbitrary point in S^2. Then, in $S^2 \times S^1$, $\{p\} \times S^1$ is a simple closed curve. Draw a schematic diagram of this. Call this simple closed curve C. Does C bound a disk in $S^2 \times S^1$?

   (c) What familiar space is $(S^2 \times S^1) - (N_\epsilon(C))^\circ$ (i.e., $(S^2 \times S^1)$ minus the interior of an ϵ-neighborhood of C) homeomorphic to?

5. (a) It is possible to travel in $\mathbb{R}^3$ from the point $(-1, 0, 0)$ to the point $(1, 0, 0)$ by walking along straight line segments and without ever touching the y-axis. Explain how. How about without ever touching the yz-plane?

   (b) It is possible to travel in $\mathbb{R}^4$ from the point $(-1, 0, 0, 0)$ to the point $(1, 0, 0, 0)$ by walking along straight line segments and without ever touching the yz-plane $\{(x, y, z, w) \in \mathbb{R}^4 : x = w = 0\}$. Explain how.

6. A **2-component link** consists of two disjoint circles embedded in $\mathbb{R}^3$. For example, let $X \subset \mathbb{R}^3$ be the unit circle in the xy-plane centered at the origin, and let $Y \subset \mathbb{R}^3$ be the unit circle in the yz-plane centered at $(0, 1, 0)$. Then $X \cup Y$ is a 2-component link; in fact, it has its own name: the **Hopf link** (named after a mathematician). Draw a picture of this link, showing the three axes in $\mathbb{R}^3$ and the coordinates of the circles' centers. The two components of the link, X and Y, cannot be "pulled apart"; more precisely, if we let Y' be a unit circle centered at $(0, 5, 0)$, then $X \cup Y$ is not isotopic to $X \cup Y'$. This is not easy to prove rigorously with what we know so far, but should be clear intuitively—do you see it?

   Now, $\mathbb{R}^3$ can be viewed as a subset of $\mathbb{R}^4$: $\mathbb{R}^3 = \{(x, y, z, w) \in \mathbb{R}^4 : w = 0\}$. Then $X \cup Y \subset \mathbb{R}^3 \subset \mathbb{R}^4$. Explain informally how $X \cup Y$ can be "pulled apart" in $\mathbb{R}^4$.

7. Give a detailed and rigorous proof that if $n > m$, then S^n cannot be embedded in $\mathbb{R}^m$. (Use the theorem that says S^n cannot be embedded in $\mathbb{R}^n$.)

8. Prove or disprove: If two connected topological spaces are glued together, the result is connected. Here's a precise statement: Let X and Y be connected topological spaces. Let $\sim$ be an equivalence relation on $X \cup Y$ such that for at least one point $x \in X$ and at least one point $y \in Y$, $x \sim y$; then $(X \cup Y)/\sim$ is connected.

Extra Credit Problems

9. (a) Explain how $S^2 \times S^1$ can be viewed as two solid tori glued together along their boundaries.
   (b) Explain how S^3 can be viewed as two solid tori glued together along their boundaries. Hint: view it as the boundary of $B^2 \times B^2$ ($\simeq B^4$).
   (c) Explain why the above does *not* imply $S^3 \simeq S^2 \times S^1$.

10. Give an embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$.