Definition 1. A subset D of a topological space X is **dense** if every open subset of X intersects D.

1. Give a countable dense subset of $\mathbb{R}$.

2. Let f and g be continuous functions from a topological space X to a Hausdorff topological space Y such that they agree on a dense subset $D \subset X$, i.e., $f|D = g|D$. Prove $f = g$.

3. (The Cantor function, a.k.a. Cantor-Lebesgue Function) Construct a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$, $f(1) = 1$, and f is constant on every interval in the complement of the Cantor set. (Is your function differentiable?)

The following problems are a mix of Topology and Analysis (measure theory).

4. Find a dense subset $D \subset \mathbb{R}$ with **measure zero**, i.e., $\forall \epsilon > 0$ there is a set of open intervals such that their union contains D and the sum of their lengths is less than ϵ.

5. Prove the Cantor set is uncountable. Prove the complement of the Cantor set in $[0, 1]$ has "full measure", i.e., the sum of the lengths of the intervals that make up the complement of the Cantor set in $[0, 1]$ is 1.

   Conclusion: $\mathbb{R}$ has an uncountable subset with measure zero.