Suppose we were 2-dimensional creatures, living in $\mathbb{R}^2$. If a "2D Christopher Columbus" set out sail and traveled in a "constant direction", he would never come back home again. On the other hand, if the world were a closed surface (such as S^2 or T^2), then traveling in a "constant direction" might eventually get him back home.

Is the same possible for the 3D universe we live in? Is it possible that traveling in a fast spaceship along a "constant direction" for a long time might get us back to our starting point? *Can you think of any closed (compact, no boundary) 3-manifolds*?

Definition 1. For $n \geq 0$, the n**-sphere** is defined as: $S^n = \{\vec{x} \in \mathbb{R}^{n+1} \mid d(\vec{x}, \vec{0}) = 1\}$.

Theorem 1. For $n \geq 1$, S^n cannot be embedded in $\mathbb{R}^n$.

Example 1. According to the above theorem, can S^5 be embedded in $\mathbb{R}^3$? [1]

A "flatlander" (a 2D person who lives in "flatland", i.e., a 2D world) cannot really visualize a 2-sphere, since a 2-sphere can not be embedded in $\mathbb{R}^2$. Similarly, we, who live in a world that *locally* looks like $\mathbb{R}^3$, cannot visualize a 3-sphere, even though we might very well be living in one! But we can learn to work with it — and with many other mathematical objects that we cannot visualize — by learning from flatlanders!

Example 2. One of several ways a flatlander can think of a 2-sphere is: two closed disks glued along their boundaries. Similarly, we can think of a 3-sphere as two closed 3-balls (i.e., 3-dimensional balls in $\mathbb{R}^3$) glued along their boundaries.

Q: *Closed* has two meanings: one for topological subspaces, one for manifolds. Which one do we mean when we say *closed ball*? [2]

Q: How should $\sim$ be defined so that $S^3 \simeq [\overline{B_1(0,0,0)} \cup \overline{B_1(5,0,0)}]/\sim$? [3] We often write this as $\overline{B_1(0,0,0)} \cup_\partial \overline{B_1(5,0,0)}$. It means we're gluing the two balls along their boundaries.

Example 3. Let's try to see why the above description of S^3 is consistent with the formal definition given at the beginning. In other words, we'd like to find a homeomorphism between $S^3 = \{\vec{x} \in \mathbb{R}^4 \mid d(\vec{x}, \vec{0}) = 1\}$ and $\overline{B_1(0,0,0)} \cup_\partial \overline{B_1(5,0,0)}$.

Let's first do it in one dimension lower. Here's how a flatlander might describe a homeomorphism between $S^2 = \{\vec{x} \in \mathbb{R}^3 \mid d(\vec{x}, \vec{0}) = 1\}$ and $\overline{B_1(0,0)} \cup_\partial \overline{B_1(5,0)}$:

(1) Send the North Pole $(0,0,1) \in S^2 \subset \mathbb{R}^3$ to the point $(0,0) \in \overline{B_1(0,0)}$. (2) Send the South Pole $(0,0,-1) \in S^2 \subset \mathbb{R}^3$ to the point $(5,0) \in \overline{B_1(5,0)}$. (3) Send the Equator $\{(x,y,0) \in \mathbb{R}^3 \mid x^2+y^2 = 1\} \subset S^2 \subset \mathbb{R}^3$ to $\partial(\overline{B_1(0,0)}) = \partial(\overline{B_1(5,0)})$. (4) Send circles in the Northern Hemisphere parallel to the Equator to circles in $\overline{B_1(0,0)}$ centered at $(0,0)$. (5) Send circles in the Southern Hemisphere parallel to the Equator to circles in $\overline{B_1(5,0)}$ centered at $(5,0)$.

Q: What is the intersection of S^2 with the horizontal plane of height $1/2$ in $\mathbb{R}^3$? How about heights 0, 1, -1, 2? [4] These are called horizontal **cross sections** of S^2.

Q: How would you rigorously define a **horizontal hyperplane** in $\mathbb{R}^4$ (i.e., a "horizontal $\mathbb{R}^3$" in $\mathbb{R}^4$)? [5]

Q: What is the intersection of $S^3 \subset \mathbb{R}^4$ with each of the following hyperplanes: (a) $\{(x,y,z,w) \in \mathbb{R}^4 \mid w = 0\}$. (b) $\{(x,y,z,w) \in \mathbb{R}^4 \mid w = 1\}$. (c) $\{(x,y,z,w) \in \mathbb{R}^4 \mid w = 1/2\}$. [6]

[1] No. Although the theorem doesn't say this directly, it does imply it. Proof: Homwork.
[2] We mean "topological"; i.e., a closed subset of $\mathbb{R}^3$.
[3] $\{(x,y,z) \sim (x+5,y,z)\}$.
[4] A horizonal circle of radius $\sqrt{3}/2$. A horizontal great circle (i.e, of radius 1); a point; a point; ϕ.
[5] $\{(x,y,z,w) \in \mathbb{R}^4 \mid w \text{ is a constant}\}$.
[6] A great 2-sphere. A point. A (not great) 2-sphere.

Q: Now try using horizontal cross sections of S^3 to show S^3 is homeomorphic to $\overline{B_1(0,0,0)} \cup_\partial \overline{B_1(5,0,0)}$.

Definition 2. Let X be a topological space. An embedded circle in X is called a **simple closed curve** (scc). (*Simple* means not self-intersecting; *closed* means it's a loop – no endpoints.)

Definition 3. Let X be a topological space, with $A \subset B \subset X$. We say A **bounds** B iff $A = \partial B$.

Example 4. The unit circle $S^1 \subset \mathbb{R}^2$ bounds the unit disk $\overline{B_1(0,0)} \subset \mathbb{R}^2$.

Example 5. For 3D beings like us, it is easy to *see* that every scc C in the 2-sphere bounds a (not necessarily round) disk *on each side*; i.e., there exist two embedded closed disks $D_1, D_2 \subset S^2$ with disjoint interiors such that $C = \partial D_1 = \partial D_2$. (Although "easy to see", this is rather difficult to prove rigorously. It's called the Jordan Curve Theorem.) However, for a flatlander, who thinks of S^2 as $\overline{B_1(0,0)} \cup_\partial \overline{B_1(5,0)}$, this is not as easy to see. Let $C \subset \overline{B_1(0,0)}$ be the circle of radius 1/4 around the point $(1/2, 0)$. Draw a picture and shade in each of the two disks that C bounds in $\overline{B_1(0,0)} \cup_\partial \overline{B_1(5,0)}$.

Example 6. Now repeat the above example in one dimension higher: try to see why an embedded S^2 in S^3 bounds a closed 3-ball on *each* side.

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Example 7. To work with $S^2 \times S^1$, it is often helpful to think of it as $S^2 \times I$ with $S^2 \times \{0\}$ glued to $S^2 \times \{1\}$.

Q: Is $S^2 \times S^1$ a manifold? If so, is it a closed manifold? If not, why not? [7]

Q: Find a 2-sphere in $S^2 \times S^1$ that does not bound a ball on either side. [8]

Q: Find a 2-sphere in $S^2 \times S^1$ that bounds a ball on one side only. [9]

Q: Is there a 2-sphere in $S^2 \times S^1$ that bounds a ball on both sides? [10]

Theorem 2. The only connected 3-manifold in which an embedded 2-sphere bounds a ball on both sides is the 3-sphere.

Proof: (Idea) If an embedded 2-sphere bounds a ball on both sides, then the two balls share the same boundary. But we already saw above that two balls glued along their boundaries yields an S^3.

Theorem 3. $S^3 \not\cong S^2 \times S^1$. Proof: Homework.

Example 8. Recall the definition of the connected sum of two n-manifolds, M and N: remove an open n-ball from each manifold; then $M - B_1$ and $N - B_2$ will each have a boundary component homeomorphic to S^{n-1}. Glue these boundaries together to obtain the connected sum of M and N.

Q: What is the connected sum of an arbitrary surface with a 2-sphere? Why? What is the connected sum of an arbitrary 3-mfd with a 3-sphere? Why? [11]

[7] It's a closed manifold.
[8] $S^2 \times \{x\}$, where x is any point in S^1.
[9] Take the boundary of any closed ball in $S^2 \times S^1$.
[10] No, by the next theorem.
[11] For both, the connected sum is homeomorphic to the original manifold.