We will soon see the classification of 2-manifolds; but before doing so, we need to learn about embeddings, isotopy, and connected sums. Embeddings and connected sums will be defined precisely. A formal definition of isotopy is more involved, however, and will be given later. But because it is a rather intuitive and easy to understand concept, we will define it informally here, and give examples. It is a very important concept in topology; and it will be useful in the next section, where we will begin to study surfaces (*surface* is a synonym for "2-manifold").

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Definition 1. Let X and Y be topological spaces. An **embedding** (imbedding in British spelling) of X into Y is a map $f : X \to Y$ such that f is a homeomorphism from X onto its image $f(X)$, where $f(X)$ is given the subspace topology as a subset of Y.

(So, informally, a embedding is a map $f : X \to Y$ that would be a homeomorphism if it were onto.)

Example 1. Find an embedding of S^1 in the yz-plane in $\mathbb{R}^3$. Draw a picture. [1]

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Theorem 1. S^1 cannot be embedded in $\mathbb{R}$.

Proof. (Sketch only — details to be completed in HW): Let $f : S^1 \to \mathbb{R}$ be any 1-1 and continuous map. Then its image $f(S^1)$ is connected (why?). Since $f(S^1) \subset \mathbb{R}$, there exists a point $z \in f(S^1)$ such that $f(S^1) - \{z\}$ is not connected (why?). But $S^1 - f^{-1}(z)$ is connected (why?). Therefore we have a contradiction (why?). □

The above theorem holds for higher dimensions as well, but is more difficult to prove:

Theorem 2. For $m \leq n$, S^n cannot be embedded in $\mathbb{R}^m$.

Proof: Omitted.

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Informal Definition: Two embeddings of topological spaces X_1 and X_2 in $\mathbb{R}^n$ are said to be **isotopic** if one can be "deformed" into the other. The "process" of deforming one into the other is called an **isotopy**.

Example 2. Each of the following is an embedding of S^1 in $\mathbb{R}^2$. Which embeddings are isotopic to each other?

(a) (b) (c)

Note. Isotopy is *stronger* than homeomorphism; this means "isotopic" implies "homeomorphic", but not vice versa. More precisely, if two embedded topological spaces X_1 and X_2 are isotopic to each other, then they must be homeomorphic to each other (the proof of this follows immediately from the formal definition of isotopy, which we'll see later). In contrast, in the above example we see two embedded topological spaces that are homeomorphic but not isotopic.

[1] $f : S^1 \to \mathbb{R}^3$, $f(x, y) = (0, x, y)$.

Example 3. Which of the following embedded surfaces (2-manifolds) in $\mathbb{R}^3$ are homeomorphic? Which are isotopic? [2]

(a) (b) (c)

(d) (e)

Connected Sums

There is a very useful way of "connecting" or "joining" two n-manifolds to obtain a new one; it is called the *connected sum*, denoted by the symbol $\#$. We present an example before giving the definition.

Example 4. Take two tori, remove a small open disk from each, then glue the two punctured tori along their circle boundaries; the surface we obtain is informally called a *2-hole torus* (and formally called a *closed surface of genus two*).

Definition 2. Let M and N be connected n-manifolds. Let $x \in M$, $y \in N$. Let B_x and B_y be neighborhoods of x and y, respectively, such that they are homeomorphic to open balls in $\mathbb{R}^n$, and their boundaries ∂B_x and ∂B_y are homeomorphic to S^{n-1}. The quotient space obtained by gluing $M - B_x$ to $N - B_y$ along their sphere boundaries is called the **connected sum** of M and N, and is denoted by $M \# N$.

Example 5. What is the connected sum of a torus with a 2-sphere? [3]

Example 6. What is the connected sum of a p-hole torus with a q-hole torus? [4]

Example 7. This is a warm-up for our upcoming study of surfaces.

Let $X = B_1(0,0)$ be the closed unit disk in $\mathbb{R}^2$. Define an equivalence relation $\sim$ on $\partial X = S^1$ as follows: $\forall a, b \in \partial X$, $a \sim b$ iff $(a_1, a_2) = (-b_1, -b_2)$, where $a = (a_1, a_2)$, $b = (b_1, b_2)$.

Let $Y = X/\sim$. In other words, Y is obtained from X by pairwise identifying **antipodal** (=opposite) points on its boundary. Don't try to visualize this — it's not possible! Just visualize one small portion of the gluing.

Q: Is Y a manifold? [5]

Q: Can Y be embedded in $\mathbb{R}^3$? [6]

The surface Y is called the **projective plane**, denoted by $\mathbb{R}\mathrm{P}^2$.

[2] (a) and (b) are homeomorphic and isotopic to each other. (d) and (e) are homeomorphic but not isotopic. (c) isn't homeomorphic to any of them — it's homeomorphic to a torus.

[3] It's again a torus! $T^2 \# S^2 = T^2$.

[4] It's a $(p+q)$-hole torus.

[5] Yes; why?

[6] No (proof is difficult.) This is why it's impossible to really visualize Y.