There are lots of different topological spaces, some of which are very strange, counter-intuitive, and pathological, but also interesting. Some topological spaces, on the other hand, are in some sense very "nice and intuitive." We are going to focus our studies on the latter kind: manifolds. These are in a way the ones that are the least strange and the most useful (or at least the most talked about). We will need two definitions before defining what a manifold is.

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Definition 1. Let x be a point in a topological space X. Any open subset of X that contains x is called a **neighborhood** of x.

(Some authors define a neighborhood of x as any set that contains an open set that contains x. These authors use the term *open neighborhood* for what we call a neighborhood. Each approach has its own advantages and disadvantages.)

Example 1. Let $X = [0, 1] \subset \mathbb{R}$. Determine whether each of the following is a neighborhood of the point 0.3 in X (be careful: it says "in X" not "in $\mathbb{R}$").

$(0.2, 0.8)$; $[0, 1]$; $[0, .7)$; $(0.2, 0.4) \cup (0.6, 0.7)$; $[0.2, 0.4)$. [1]

Example 2. Let $X = S^1 \subset \mathbb{R}^2$. Give an example (other than S^1 itself) of a neighborhood of the point $(1, 0) \in S^1$. Ans: $\{(x, y) \in S^1 : -1/2 < y < 1/2\}$. (Draw a picture of this neighborhood for yourself!)

Recall that $(0, 1) \simeq \mathbb{R}$. One can similarly show that every open ball in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$. (Prove it!) So, in the following, whenever you read "homeomorphic to $\mathbb{R}^n$," keep in mind that it's equivalent to saying "homeomorphic to an open ball $\mathbb{R}^n$."

Definition 2. A topological space X is said to be **locally homeomorphic** to $\mathbb{R}^n$ iff every point in X has some neighborhood that is homeomorphic to $\mathbb{R}^n$.

Example 3. Let $X = S^1 \subset \mathbb{R}^2$.

Q: Is X locally homeomorphic to $\mathbb{R}$? [2] Is X homeomorphic to $\mathbb{R}$? No; proved in a previous section.

Q: Let $x \in S^1$. Is *every* neighborhood of x homeomorphic to $\mathbb{R}$? No.

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Definition 3. An n**-dimensional manifold** (or an n-manifold, for short) is a topological space that is Hausdorff, Second Countable, and locally homeomorphic to $\mathbb{R}^n$.

For the time being, don't worry about Hausdorff and Second Countable. The important part of the definition that we need to focus on now is "locally homeomorphic to $\mathbb{R}^n$."

Example 4. Is S^1 a 1-manifold? Yes, because it is locally homeomorphic to $\mathbb{R}$.

Example 5. Is a torus $T^2 = S^1 \times S^1$ a manifold? [3]

Example 6. Let $X = x$-axis $\cup$ y-axis. Give $X \subset \mathbb{R}^2$ the subspace topology.

Q: Is X a manifold? [4]

Q: Is $X - \{(0, 0)\}$ a manifold? Yes. Of what dimension? 1. Is it a connected manifold? No; it has four *components* (we'll see a precise definition of this word later).

Example 7. Every point in S^1 has a neighborhood that is homeomorphic to $\mathbb{R}$, so S^1 is a 1-manifold.

Example 8. Is $(0, 1) \subset \mathbb{R}$ a manifold? Yes, it's a 1-manifold. How about $[0, 1] \subset \mathbb{R}$? No, because the endpoints, 0 and 1, do not have neighborhoods in $[0, 1]$ that are homeomorphic to $\mathbb{R}^n$ for any n. But $[0, 1]$ is a manifold-with-boundary. We'll see a precise definition for this in the next section.

[1] Yes. Yes; why? Yes; why? Yes. No.

[2] Yes; each point has a neighborhood that's homeomorphic to an open interval; an open interval is homeomorphic to $\mathbb{R}$.

[3] Yes; every point in T^2 has a neighborhood homeomorphic to $\mathbb{R}^2$. So it's a 2-dimensional manifold.

[4] No: the origin has no neighborhood in X that's homeomorphic to $\mathbb{R}^n$ for any n.