1. Let $A \subset \mathbb{R}$. Prove rigorously that if a, b, c are real numbers such that $a < b < c$, where $a, c \in A$ and $b \notin A$, then A is not connected.

2. (Intermediate Value Theorem) Let X be a connected topological space, and $f : X \to \mathbb{R}$ a continuous map. Show that for all $x, z \in X$, if $f(x) \leq f(z)$, then $[f(x), f(z)] \subseteq f(X)$.

3. (Brouwer Fixed Point Theorem, Dimension 1) Let $f : I \to I$ be a continuous map. Prove that f has a **fixed point**, i.e., $\exists x \in I$ such that $f(x) = x$. Hint: Define $g(x) = f(x) - x$, and apply the previous problem to g.

4. Find a continuous map $f : (0, 1) \to (0, 1)$ with no fixed points.

5. Suppose you hike up a trail starting at 8:00 AM on Saturday, camp out Saturday night, hike back down the same trail, starting at 8:00 AM on Sunday, and reach the bottom sometime in the afternoon. Prove there is a point on the trail such that you pass it at exactly the same time on both days. Assume that you always stay on the trail. But there are no other assumptions; e.g, you may stop and take a break at various times, or even retrace part of the trail to look for something that you think you may have dropped earlier!

6. (Borsuk-Ulam Theorem, Dimension 1) Suppose you have a disk-shaped garden. You build a fence on its circle boundary. The height of the fence may not be constant, but is continuous. Prove there is a pair of antipodal (opposite) points on the circle at which the fence has the same height.

7. Can the floor of a room be so uneven that no matter where you put a four-legged table on it, it will always wobble? Assume the floor's surface is continuous, the table's legs have the same length, and their tips form a square.

8. *Theorem* (Borsuk-Ulam Theorem, Dimension 2) Let $f : S^2 \to \mathbb{R}^2$ be continuous. Then there exist antipodal points $p, -p \in S^2$ such that $f(p) = f(-p)$.

   Use the above theorem (without proof) to show: At any given moment, there exist two antipodal points on the surface of Earth that have the same temperature and pressure. What assumptions, if any, do you need to make?

9. Intuitively it is very plausible that the identity map $f : S^1 \to S^1$ is not homotopic to a constant map. Taking this for granted, prove that there is no **retraction** from the closed unit disk D^2 to the unit circle S^1, i.e., there is no continuous map $g : D^2 \to S^1$ such that $g|_{S^1} =$ identity.

10. Use the above to show that every continuous map $f : D^2 \to D^2$ has a fixed point.

11. Similar to the above, we have:

    Theorem (Brouwer Fixed Point Theorem, Dimension n) Every continuous map $f : \overline{B^n} \to \overline{B^n}$ has a fixed point.

    Use this theorem (without proof) to show: No matter how you stir a jar of honey, there is some molecule that will end up in the same place it was before the stirring. (We're assuming "continuity in the honey": there are infinitely many molecules, and the closer two molecules are to each other before the stirring, the closer they will be after the stirring.)

12. Suppose we ask n couples to stand on two concentric and "nearby" circles. The men are to stand on the outer circle, ordered clockwise alphabetically according to their last names, the women on the inner circle, ordered counterclockwise alphabetically according to their husbands' last names. Suppose each couple gets one dollar if the husband and the wife are standing next to each other, fifty cents if they are one person apart, and nothing otherwise. Prove that (even after rotating

each circle a different amount) all the couples together earn a total of exactly two dollars. Can you think of a continuous version of this (discrete) problem (in terms of maps of circles)?

Extra Credit Problems

13. Suppose you have two pancakes of arbitrary shape next to each other on a tray. Prove that it is possible, with only one straight cut, to simultaneously divide each pancake into two equal halves (measured by area).