*Definition* Let $X$ be a topological space. A loop whose image is just one point in $X$ is called a **trivial loop**. A loop is said to be **null-homotopic** iff it is loop-homotopic to a trivial loop in $X$. (For a definition of *loop-homotopic* see the previous set of homework problems.)

1. (a) Show that every loop in $\mathbb{R}^2$ is null-homotopic.
   (b) Show that any two loops in $\mathbb{R}^2$ are null-homotopic to each other. Do this twice, using each method below once.
   Method 1: Use Part (a) above, and the fact that being loop-homotopic is an equivalence relation. (We've shown that "being homotopic" and "being isotopic" are equivalence relations. We haven't shown that "being loop-homotopic" is an equivalence relation, but you may use this fact; it's proof is very similar to that of the former two.)
   Method 2: Construct a loop-homotopy. (Don't forget to show that in every "frame" of your loop-homotopy you in fact see a loop.)
2. (a) Draw three loops on the torus, $T^2$, such that no two of them are loop-homotopic to each other. (No proof necessary). Can you find more than three?
   (b) How many loops are there on the annulus $S^1 \times I$ such that no two of them are loop-homotopic to each other? Support your answer by constructing the loops (but you don't need to prove that no two of them are loop-homotopic to each other).
3. Recall that $\mathbb{RP}^2$ is defined as (this is one of two definitions we have seen): the closed unit disk with **antipodal** (i.e., opposite) points on its boundary identified; $\mathbb{RP}^2 = D^2/\{\forall x \in \partial D^2,\ x \sim -x\}$. Let $q : D^2 \to \mathbb{RP}^2$ be the quotient map.
   (a) Let $A$ be the horizontal diagonal in $D^2$, i.e., $A = \{(x, y) \in D^2 : y = 0\}$. Let $\alpha = q(A) \subset \mathbb{RP}^2$. Then $\alpha$ is a closed curve in $\mathbb{RP}^2$. Why? Technically, $\alpha$ is not really a loop. Why?
   (b) Give a homeomorphism $h$ from $I$ to $A$.
   (c) Explain why the composition $q|_A \circ h$ is a loop in $\mathbb{RP}^2$. What is the image of this loop? Do you think this loop is null-homotopic (just Y or N, without proof)?
   (d) Define $g : I \to D^2$ by $g(s) = (\cos(2\pi s), \sin(2\pi s))$. Then $q|_{g(I)} \circ g$ is a loop in $\mathbb{RP}^2$. Why? Prove that the loop $q|_{g(I)} \circ g$ is null-homotopic.
4. How many loops are there on $\mathbb{RP}^2$ such that no two of them are loop-homotopic to each other? (Proof not necessary.)
5. Prove that the Möbius band $M$ is not orientable: give an embedding $h : D^2 \to M$ and an isotopy between $h$ and its mirror image.

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Extra Credit Problems

6. (a) Represent $3T^2$ as a polygon with edges identified appropriately.
   (b) Represent $3\mathbb{RP}^2$ as a polygon with edges identified appropriately.
7. Prove that a continuous function $f : X \to Y$ whose domain is a compact metric space $(X, d_X)$ is **uniformly continuous**, i.e., $\forall \epsilon > 0$, $\exists \delta$ such that $\forall a, b \in X$, $d_X(a, b) < \delta \Rightarrow d_Y(f(a), f(b)) < \epsilon$.
8. Use Sperner's Lemma (see Wikipedia) and the above to prove there is no retraction from a disk to its boundary, i.e., there is no continuous map $f : D^2 \to S^1$ that is identity on $S^1$.
9. Use the above to prove every continuous map from the disk to itself has a fixed point.