1. Give a precise definition of what it means for a path or a loop to be *self-intersecting.*

2. Let $g : I \to \mathbb{R}^2$ be given by $g(x) = (\cos(2\pi x), \sin(2\pi x))$. Let $h : I \to \mathbb{R}^2$ be given by $h(x) = (\cos(\pi x), \sin(\pi x))$.

   (a) Draw the image of each of g and h.

   (b) Only one of the following two questions is valid. Answer it (with proof), and then explain what is wrong with the other question.

      i. Is g isotopic to h in $\mathbb{R}^2$?

      ii. Is g homotopic to h in $\mathbb{R}^2$?

   (c) Let $j : I \to \mathbb{R}^2$ be given by $j(x) = (3\cos(2\pi x), 3\sin(2\pi x))$. Intuitively, we'd like to think of g and j as isotopic loops; but they're not. So let's make the following definition:

   Definition Two simple loops $f_0 : I \to X$ and $f_1 : I \to X$ are **loop-isotopic** in X if there is a continuous map $H : I \times I \to X$ such that $H_0 = f_0$, $H_1 = f_1$, and for all $t \in I$, $H_t : I \to X$ is a simple loop.

   Prove that g and j, as defined above, are loop-isotopic in $\mathbb{R}^2$.

3. Let $Y = \overline{B_5(0,0)} - B_1(0,0) \subset \mathbb{R}^2$.

   (a) Let $g : I \to Y$ be given by $g(x) = (4,0)$, $h : I \to Y$ be given by $h(x) = (3,0) + (\cos(2\pi x), \sin(2\pi x))$. Prove that $g \sim h$ in Y.

   (b) Let $j : I \to Y$ be given by $j(x) = (-3,0) + (\cos(2\pi x), \sin(2\pi x))$. Prove that h and j are loop-isotopic in Y.

4. (a) How would you define what it means for two (not-necessarily simple) loops to be *loop-homotopic*?

   (b) Let $Y = \overline{B_5(0,0)} - B_1(0,0) \subset \mathbb{R}^2$. Define $k : I \to Y$ by $k(x) = 3(\cos(2\pi x), \sin(2\pi x))$ and $l : I \to Y$ by $l(x) = 3(\cos(4\pi x), \sin(4\pi x))$. Are k and l loop-homotopic in Y? Formal proof not necessary; but explain your reasoning.

5. (a) Prove that $\approx$ is an equivalence relation for maps.

   Hint: To prove transitivity, proceed as follows. Let f, g, and h be embeddings of X into Y, such that $f \approx g$ and $g \approx h$. To prove $f \approx h$, we'd like to find a map H from what to what, such that what?

   By definition, $f \approx g$ means there is an isotopy $F : X \times I \to Y$ from f to g. Similarly, there is an isotopy $G : X \times I \to Y$ from g to h.

   First, define a map $J : X \times [0,2] \to Y$ as follows: $J(x,t) = \begin{cases} F(x,t) & \text{if } 0 \le t \le 1 \\ G(x,t-1) & \text{if } 1 < t \le 2 \end{cases}$

   Then let $H(x,t) = J(x, 2t)$, and show that H is the desired isotopy.

   Don't forget to also show that $\approx$ is reflexive and symmetric.

   (b) It is also true that $\sim$ is an equivalence relation. In your proof above, what would you need to change, and what would you keep the same, in order to prove this?