1. Let $A = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x, y, z \leq 1$ and at least two of x, y, z are in the set $\{0,1\}$ $\}$. Let $F = \partial(\overline{N_{0.1}(X)})$. Draw a picture of A and a picture of F. Show, using pictures, that $F \simeq nT^2$ for some n.

2. Let (X, d) be a metric space, and let $A \subset X$. True or False: $\forall \epsilon > 0$, $N_\epsilon(A) = \bigcup_{a \in A} B_\epsilon(a)$. Prove your answer.

3. Take a strip of paper and glue its two shorter edges in such a way as to create a Möbius band. In the following, let $M = [0, 1]^2/\{(0, y) \sim (1, 1 - y)\}$ denote the Möbius band, and let $q : [0, 1]^2 \to M$ be the corresponding quotient map. Let $A = [0, 1] \times \{1/2\} \subset [0, 1]^2$.

   (a) Explain why $q(A) \subset M$ is a circle. Draw this circle on the Möbius band that you created. We call this circle the **core** of the Möbius band.

   (b) Use a pair of scissors to cut your Möbius band along its core. Is this a separating circle? Use schematic diagrams to prove your answer.

   (c) Let $B = [0, 1] \times \{1/4\} \cup [0, 1] \times \{3/4\}$. Is $q(B)$ a circle? Use diagrams to explain.

   (d) Create a new Möbius band, and draw $q(B)$ on it. *Before* cutting along it, draw diagrams to try to predict whether or not $M - q(B)$ is connected. Now cut. Was your prediction correct? Draw diagrams to explain why $q(B)$ does or does not separate M.

4. Let $f : X \to Y$ be a homeomorphism between topological spaces. Let $A \subset X$. True or false: A separates X iff $f(A)$ separates Y. Prove your answer.

5. Prove the torus and the Klein Bottle are not homeomorphic. Hint: use some of the theorems of Section 9.

6. Prove, without using Theorem 2 of Section 9, that $\mathbb{R}\mathrm{P}^2 \not\simeq S^2$. Hint: (i) *Jordan Curve Theorem*: Every embedded circle $C \subset S^2$ separates S^2 (you may use this without proving it). (ii) Show there exists a non-separating circle $C \subset \mathbb{R}\mathrm{P}^2$ (from above we know that the Möbius band has a non-separating circle). (iii) Assume $\mathbb{R}\mathrm{P}^2 \simeq S^2$, and use (i) and (ii) to get a contradiction.

7. (a) Prove that gluing two Möbius bands along their circle boundaries yields a Klein Bottle: $M \cup_\partial M \simeq K$. (You may find it easier to prove that a Klein Bottle can be cut up into two Möbius bands.)

   (b) Use Part (a) above to prove that the Klein Bottle is homeomorphic to the connected sum of two projective planes.

   (c) Use the Extra Credit Problem below to prove that $T^2\#\mathbb{R}\mathrm{P}^2 \simeq \mathrm{K}\#\mathbb{R}\mathrm{P}^2$.

8. Use the fact that $\mathbb{R}$ is connected to prove that S^1 is connected.

Extra Credit Problems

9. Prove that $T^2\#\mathbb{R}\mathrm{P}^2 \simeq 3\mathbb{R}\mathrm{P}^2$.

10. Embed the Möbius band (by drawing a picture of it) in $\mathbb{R}^3$ in such a way that its boundary is the unit circle in the xy-plane.
