1. State which of the following spaces are homeomorphic to each other. You do not need to prove your answers; but give brief explanations or draw pictures (or both) to support them.
   (a) $X = \overline{B_1(0,0)} \subset \mathbb{R}^2$ (i.e., X is the closed unit disk in the plane).
   (b) $Y = X - B_{0.5}(0,0)$. (c) Y° (the interior of Y, where Y is viewed as a subspace of $\mathbb{R}^2$).
   (d) $X/\partial X$. (This means identify the whole boundary of X into one point – recall that $\partial X = S^1$.)
   (e) $S^1 \times [0,1]$. (f) $S^1 \times (0,1)$. (g) $S^1 \times [0,1)$. (h) $S^1 \times \mathbb{R}$. (i) $\mathbb{R}^2$
   (j) $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. (k) $S^2 - \{(1,0,0)\}$. (l) $S^2 - \{(0,0,1),(0,0,-1)\}$.
   (m) $S^2 - B_{0.5}(0,0,1)$.

2. Which of the following letters, considered as subspaces of $\mathbb{R}^2$, are manifolds? Give a brief explanation for each one that is not a manifold.

   A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

3. How many connected non-homeomorphic 1-manifolds do you think there are? List as many as you can think of.

4. Is an open ball minus its center a manifold? More precisely, let $x \in \mathbb{R}^n$, $r > 0$. Is $B_r(x) - \{x\}$ a manifold? Prove your answer.

5. (a) Is $S^1/\{(1,0) \sim (-1,0)\}$ a manifold? Explain.
   (b) Is $S^1/\{(x,y) \sim (-x,-y)\}$ a manifold? Explain.

Extra Credit Problems

Definition A **component** of a topological space X (sometimes also called a *connected component*, for emphasis) is a connected subset of X that is not a proper subset of any other connected subset of X (in short, it's a maximal connected subset of X).

6. Let X be a topological space that has finitely many components. Prove that every component of X is open and closed.

7. Given a topological space X, define a relation $\sim$ on X by $x \sim y$ iff X contains a connected subset that contains both x and y.
   (a) Prove that $\sim$ is an equivalence relation. (Just prove transitivity. It's obvious that $\sim$ is reflexive and symmetric.)
   (b) Prove that the components of X are the equivalence classes of $\sim$.

8. *Definition* A topological space X is path connected if $\forall x, y \in X$ there is a **path** from x to y; i.e., there is a continuous map $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$.

   Prove or disprove each of the following.

   (a) Every connected space is path connected. (Tedious details may be omitted.)
   (b) Every path connected space is connected.

   Hint: For one of the above, consider the following subset $X \subset \mathbb{R}^2$, with the subspace topology inherited from $\mathbb{R}^2$; it is called the *topologist's sine curve.*

$$X = \{(x,y) \in \mathbb{R}^2 : (x = 0 \text{ and } -1 \le y \le 1) \text{ or } (x > 0 \text{ and } y = \sin(1/x))\}$$