1. Show by example that the union of an infinite collection of closed subsets of a metric space is not necessarily closed.

2. In a previous homework problem we showed that the union of any two closed subsets of a metric space is closed. Use this fact, and mathematical induction, to prove that the union of any finite collection of closed subsets of a metric space is closed.

3. In a previous homework problem we showed that the union of any collection of open subsets of a metric space is open. Use this fact to prove that the intersection of any collection of closed subsets of a metric space is closed.

   Hint: Use de Morgan's Law: Let $\{A_\alpha \mid \alpha \in I\}$ be any collection of subsets of some fixed set, where I is an index set; then

$$\left(\bigcap_{\alpha \in I} A_\alpha \right)^c = \bigcup_{\alpha \in I} {A_\alpha}^c.$$

Summary of unions and intersections of open or closed subsets

Union of any collection of open sets is open.

Intersection of any finite number of open sets is open.

Intersection of an infinite number of open sets may or may not be open.

Intersection of any collection of closed sets is closed.

Union of any finite number of closed sets is closed.

Union of an infinite number of closed sets may or may not be closed.

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4. Let $A \subseteq \mathbb{R}^2$ be given by: $A = \{(x, y) \mid 1 < x^2 + y^2 \leq 2\} \cup \{(0, 0)\}$.

   (a) Draw a picture of A.

   (b) Draw a picture of A° (the interior of A) and use set notation to describe it.

   (c) Draw a picture of $\overline{A}$ (the closure of A) and use set notation to describe it.

   (d) Draw a picture of ∂A (the boundary of A) and use set notation to describe it.

   (e) Find all limit points of A that are not in A. Is $(0, 0)$ a limit point of A? Why?

Extra Credit Problems

5. Let A be a subset of a metric space M. Prove A° is open.

6. Let A be a subset of a metric space M. Prove A° is the union of all subsets $C \subseteq A$ which are open in M. Prove $\overline{A}$ is the intersection of all subsets C such that $A \subseteq C$ and C is closed in M.