Mathematical World • Volume 4

Intuitive Topology

V. V. Prasolov

Translated from the Russian by
A. Sossinsky

American Mathematical Society

В. В. ПРАСОЛОВ
НАГЛЯДНАЯ ТОПОЛОГИЯ

Translated from an original Russian manuscript by A. Sossinsky

1991 *Mathematics Subject Classification.* Primary 54–01;
Secondary 57–01.

ABSTRACT. The book is an introductory course in topology. It is written in a rather nontraditional manner, starting with describing the main notions in a tangible and perceptible manner, and then progressing to more precise and rigorous definitions and results, reaching the level of fairly sophisticated (although completely understandable) proofs. This approach allows the author to tackle from the very outset meaningful and interesting problems, presenting examples of nontrivial and often unexpected topological phenomena.

Another nontraditional feature of the book is that it deals mainly with constructions of objects (like surfaces, knots, and links in space) and maps between these objects, rather than with general theorems implying that certain maps do not exist. To help understand the constructions, the book is supplied with numerous illustrations, which, sometimes, are more important than the text (which is then little more than a commentary).

Each chapter contains numerous problems, which are an integral part of the exposition. The solutions of problems are presented at the end of the corresponding chapter.

The book will interest any reader who has some feeling for the visual elegance of geometry and topology, including advanced students and mathematics teachers in high schools, as well as college undergraduates majoring in mathematics.

Library of Congress Cataloging-in-Publication Data

Prasolov, V. V. (Viktor Vasil′evich)
Intuitive topology/V. V. Prasolov; translated from the Russian by A. Sossinsky.
p. cm. — (Mathematical world; v. 4)
Includes bibliographical references and index.
ISBN 0-8218-0356-5 (acid-free)
1. Topology. I. Title. II. Series.
QA611.13.P73 1994
514—dc20 94-23133
CIP

The paper used in this book is acid-free and falls within the guidelines established to ensure permanence and durability.
Printed on recycled paper.
This volume was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX,
the American Mathematical Society's TEX macro system.

10 9 8 7 6 5 4 3 2 1 00 99 98 97 96 95

Foreword

Topology studies the properties of geometrical objects that remain unchanged under transformations called homeomorphisms and deformations. The initial aquaintance with this field is hindered by the fact that rigorous definitions of even the simplest notions of topology are rather abstract or very technical. For this reason the first really meaningful (and in fact readily understandable) topological theorems appear only after tedious preliminaries have been overcome. This preliminary work is mostly devoted to the detailed and accurate proofs of intuitively obvious statements: admittedly not a very exciting activity.

This book is an introductory course in topology of rather untraditional structure. We begin by defining the main notions in a tangible and perceptible way, on an everyday level, and as we go along we progressively make them more precise and rigorous, reaching the level of fairly sophisticated proofs. This allows us to tackle meaningful problems from the very outset with some success.

Another unusual trait of this book is that it deals mainly with constructions and maps (of surfaces, knots, and links in space), rather than with proofs of general theorems implying that certain maps and constructions don't exist. Such proofs, usually based on complicated invariants (e.g., so-called homotopy and homology functors), are in fact a more traditional activity for topologists, but are not the main subject matter of this book. We do consider some invariants, but only simple and effective ones.

The (numerous) illustrations are essential. In many parts of the book they are more important than the text, which is then little more than a commentary to the pictures.

In the study of mathematics, problem solving plays a crucial role. Reading ready-made proofs of theorems is a poor substitute for trying to prove them on your own. Many statements that the reader can profitably think about himself appear in the form of problems. These problems are an inherent part of our exposition, and therefore their solutions are presented at the end of each section.

A bibliography, mainly consisting of books that we recommend for the further study of topology, appears at the end of the book. Among the books and articles that had the greatest influence on this book, I would like to name the book by Rolphsen [3] and the article by Viro [5].

As is usually done in mathematical books and papers, the symbol □ marks the end of the proof of a proposition or a theorem.

It should be mentioned that the present text is based on a series of lectures given by the author in the academic year 1990–1991 to students of High School no. 57 in Moscow.

I am grateful to N. M. Fleischer for useful discussions of the manuscript.

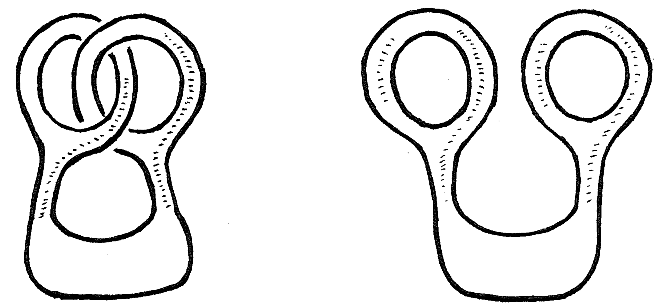
1

Deformations

Our first look at topology will involve some problems about the deformation of elastic bodies and surfaces. We shall assume that the objects considered are made from a very elastic material: their shape may be changed at will, you can bend, distort, stretch, and compress them as much as you like, but of course you may not tear them or glue parts of them together.

The deformations that you will be asked to find will seem impossible at first glance. But actually they are not difficult to visualize, as you can verify by reading their description in the solution section. However, we emphatically suggest that you try to find the solution on your own before looking at our answers.

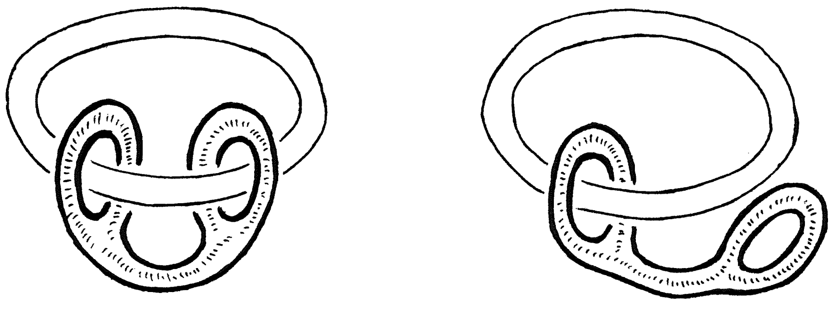
Problem 1.1. Show that the elastic body represented in Figure 1.1 (a) can be deformed so as to become the one shown in Figure 1.1 (b). In other words, were the human body elastic enough, after making linked rings with your index fingers and thumbs, you could move your hands apart without separating the joined fingertips.



(a) (b)

FIGURE 1.1

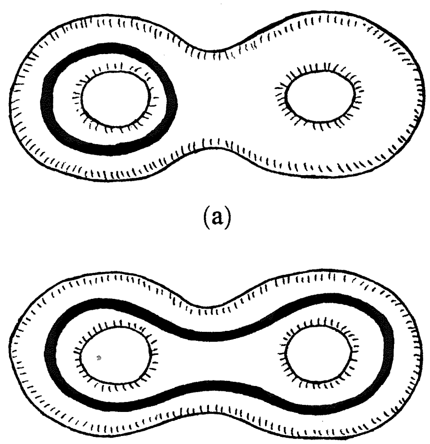
Problem 1.2. A pretzel has two holes that "hold" a doughnut (see Figure 1.2 (a)). Show that the pretzel can be deformed in such a way that one of its "handles" will unlink itself from the doughnut (Figure 1.2 (b)).



(a) (b)

FIGURE 1.2

Problem 1.3. A circle is drawn on a pretzel with two holes (Figure 1.3 (a)). Show that it is possible to deform the pretzel so that the circle will be in the position represented in Figure 1.3 (b).



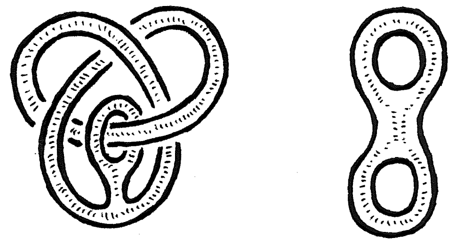
(a)

(b)

FIGURE 1.3

Problem 1.4. Show that a punctured tube from a bicycle tire can be turned inside out. (More precisely, this would be possible if the rubber from which the tube is made were elastic enough. In real life it is impossible to turn a punctured tube inside out.)

Problem 1.5. Show that the fancy pretzel represented in Figure 1.4 (a) can be deformed into the ordinary pretzel with two holes (Figure 1.4 (b)).



(a) (b)

FIGURE 1.4

Solutions.

We present most of the solutions to the problems in this book by means of pictures, which, as a rule, are self-explanatory. We sometimes indicate by arrows on the pictures the direction of motion or of deformation.

1.1. See Figure 1.5. We shall return to this deformation in §4.

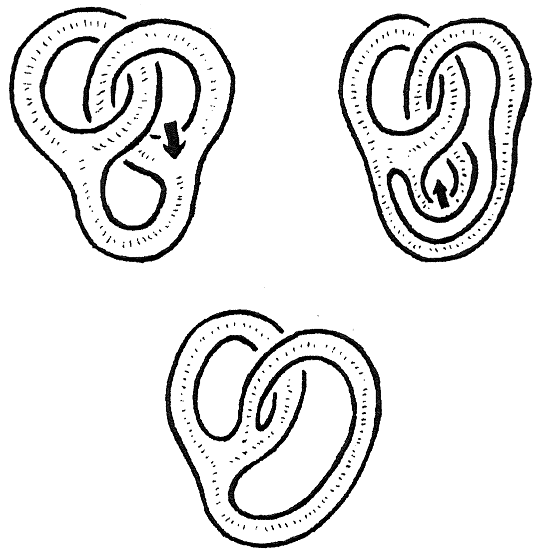


FIGURE 1.5