We are finally ready to see a precise definition of isotopy. Recall the informal definition: If X_1 and X_2 are subsets of a topological space Y, then X_1 is isotopic to X_2 iff X_1 can be continuously deformed in Y to look like X_2. The formal definition below may be difficult to understand initially and may at first seem very different from the above informal definition. Just read through it and then read the examples that follow; it'll gradually make more and more sense.

Definition 1. Let X and Y be topological spaces. We say an embedding $f : X \to Y$ is **isotopic** to another embedding $g : X \to Y$, denoted $f \approx g$, iff there exists a continuous map $H : X \times I \to Y$ such that

(1) $\forall x \in X$, $H(x, 0) = f(x)$.
(2) $\forall x \in X$, $H(x, 1) = g(x)$.
(3) $\forall t \in I$, $H(\cdot, t)$ gives an embedding of X into Y.

We say H is an **isotopy** from f to g.

The notation in condition (3) is explained further below. Here's an informal but very useful way to understand the above definition. Think of the map H as a one-minute movie. Time is represented by $t \in I$. Then,

(1) says: at time $t = 0$, we see the embedding f.
(2) says: at time $t = 1$ (the end of the movie), we see the embedding g.
(3) says: at every single instant t during the movie, we see an embedding of X into Y.

One more important feature: In our movie, the closer two frames are to each other temporally, the more alike they look. In other words, as we're watching the movie (the deformation), we should not see any "sudden jumps" in it. Which part of the formal definition corresponds to this feature? [1]

Notation: Given a fixed $t \in I$, $H(\cdot, t) : X \to Y$ denotes the map that sends each point $x \in X$ to the point $H(x, t) \in Y$. Instead of $H(\cdot, t) : X \to Y$ we often write $H_t : X \to Y$. They are equivalent.

Example 1. Let's denote the circle of radius r centered at the point $(x, y) \in \mathbb{R}^2$ by $C_r(x, y)$. (So $S^1 = C_1(0, 0)$.)

Q: Find an embedding $f : S^1 \to \mathbb{R}^2$ whose image is $C_2(0, 0)$. [2]

Q: Find an embedding $g : S^1 \to \mathbb{R}^2$ whose image is $C_3(7, 0)$.

Q: Show f is isotopic to g by following these steps:

Step 1: Find a homeomorphism $h : C_2(0, 0) \to C_3(7, 0)$.

Step 2: For an isotopy, we need to find a map H from what to what? [3]

Step 3: For $\vec{v} = (x, y) \in S^1$ and $t \in I$, let $H(\vec{v}, t) = (1 - t)f(\vec{v}) + (t)g(\vec{v})$, where f and g are thought of as vector-valued functions. Check to see if this satisfies all three conditions of the definition for the isotopy we desire. This is called a **straight-line** (or **linear**) isotopy, because during the isotopy each point "moves" in a straight line.

Theorem 1. $\approx$ is an equivalence relation.

Idea of Proof: We only give an idea why $\approx$ is transitive. You will turn this idea into a rigorous proof in homework! We have a one-minute movie in which f becomes g, and a one-minute movie in which g becomes h. We want a one-minute movie in which f becomes h. First we append the second movie to the end of the first movie. This gives us a two-minute movie in which f becomes h. Then we play this movie at twice the normal speed, and it becomes a one-minute movie, as desired.

[1] H is continuous (as a function of t).
[2] $\forall (x, y) \in S^1$, let $f(x, y) = (2x, 2y)$
[3] $H : S^1 \times I \to \mathbb{R}^2$.

Definition 2. An embedding from a circle into $\mathbb{R}^3$ is called a **knot**. An embedding from a set of (one or more) disjoint circles into $\mathbb{R}^3$ is called a **link**.

In Knot Theory (a branch of Topology), two knots or links that are isotopic to each other are considered to be equivalent. Any knot that is isotopic to $S^1 \times \{0\} \subset \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ is called **the unknot** (also called a trivial knot). (Actually, isotopy isn't strong enough; instead, one requires *ambient isotopy*, which we have not defined, and will not get into.)

Can you guess what the **unlink** with two components would be defined as? Can you draw a 2-component link that is not isotopic to the unlink?

Paths and loops

Definition 3. Let X be a topological space. A **path** in X is a continuous map $p : I \to X$. A path whose **initial point** $p(0)$ equals its **terminal point** $p(1)$ is called a **loop** or a **closed curve**. A path or a closed curve that does not intersect itself is called a **simple** path or closed curve.

Note. A path or a loop is a *map* from I to X, not just a subset of X! Intuitively, it sometimes helps to think of a path or a loop as just the *image* of the map p. It is important to remember, however, that formally it is not just the image of the map, but the map itself, that we work with.

Example 2. Determine whether each of the following maps is a path, a loop, or neither.

(a) $p : [0, 1] \to \mathbb{R}^2$, $p(t) = (t, 2t)$.

(b) $p : [0, 1] \to \mathbb{R}^2$, $p(t) = \begin{cases} (t, t) & \text{if } 0 \leq t < 1/4 \\ (1/2 - t, 1/4) & \text{if } 1/4 \leq t < 1/2 \\ (t - 1/2, 3/4 - t) & \text{if } 1/2 \leq t < 3/4 \\ (1 - t, 0) & \text{if } 3/4 \leq t \leq 1 \end{cases}$

(c) $p : [0, 1] \to \mathbb{R}^3$, $p(t) = (0, 0, 0)$. [4]

Homotopy

Definition 4. Let X and Y be topological spaces, and f and g continuous maps from X to Y. A **homotopy** from f to g is a continuous map $H : X \times I \to Y$ such that

(1) $H_0 = f$.
(2) $H_1 = g$.

We say f is **homotopic** to g, and write $f \sim g$. (Note that the only difference between a homotopy and an isotopy is the "third condition": an isotopy is a homotopy in which every H_t is an embedding.)

Example 3. T or F: If two maps are isotopic to each other, then they are homotopic to each other. How about the converse? [5]

[4] (a) Path, not loop. (b) Neither: p is not continuous. (c) Both path and Loop.
[5] T. F.