We've seen that the interval $(-1,1) = B_1(0) \subset \mathbb{R}$ is homeomorphic to $\mathbb{R}$. It is not difficult to also prove that the **open unit disk** $B_1(0,0) \subset \mathbb{R}^2$ is homeomorphic to $\mathbb{R}^2$. (In dimensions 3 or higher, we say ball; in dimension 2, disk; in dimension 1, interval or segment.)

We also showed that $[-1,1] = \overline{B_1(0)} \subset \mathbb{R}$ is not homeomorphic to $\mathbb{R}$. How? By using connectedness: $[-1,1] - \{1\}$ is connected, but there is no point x for which $\mathbb{R} - \{x\}$ is connected. Similarly, we'd like to show that the **closed unit disk** $\overline{B_1(0,0)} \subset \mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^2$. But doing this by removing one point and then considering connectedness, like before, doesn't work (can you see why?). But there is another important topological invariant that does help us prove what we want. It's called *compactness*. To define it, though, we'll first need a few other definitions.

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Definition 1. Let X be a set, and $A \subseteq X$. A collection F of subsets of X is called a **cover** of A iff

$$A \subseteq \bigcup_{B \in F} B$$

"Cover" is used both as a noun and as a verb: F is a cover of A; F covers A.

Example 1. Let $X = \mathbb{R}$, $A = [0,3]$.

Q: Does $F = \{[0,2],[1,5]\}$ cover A? Why or why not? [1]

Q: Does $F = \{[1/n,3] \mid n \in \mathbb{N}\}$ cover A? Why or why not? [2]

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Example 2. Let $X = \mathbb{R}$, $A = [0,3]$. Let $F = \{[1/n,3] \mid n \in \mathbb{N}\} \cup \{[0,1]\}$.

Q: Is F a cover of A? [3]

Q: Can you find a **finite subcover** of F, i.e., find a finite subset F' of F that covers A? [4]

Definition 2. Let X be a topological space, and F a cover of $A \subseteq X$. F is said to be an **open cover** of A if every element of F is open in X.

Example 3. Let $X = \mathbb{R}$, $A = [0,3]$.

Q: Is $F = \{[0,2],(1,5)\}$ an open cover of A? [5]

Q: Give a finite open cover of A. Ans: $F = \{(-1,2),(1,5)\}$ (or simpler: $F = \{(-1,4)\}$).

Example 4. Let $X = \mathbb{R}$, $A = \mathbb{R}$. Give a finite open cover of A. [6]

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Definition 3. A topological space X is **compact** iff every open cover of X has a finite subcover.

Example 5. Prove that $\mathbb{R}$ is not compact by finding an open cover of $\mathbb{R}$ that has no finite subcover. [7]

Example 6. Prove that $(a,b) \subseteq \mathbb{R}$ is not compact by giving an open cover for it that has no finite subcover. [8]

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[1] Yes: $A \subseteq [0,2] \cup [1,5]$.

[2] No: $A \not\subseteq \bigcup_{B \in F} B$. Why? Because 0 is not in any element of F.

[3] Yes.

[4] Let $F' = \{[0,1],[1,3]\}$.

[5] No: $[0,2]$ is not open in $\mathbb{R}$.

[6] $F = \{\mathbb{R}\}$.

[7] Let $F = \{(n,n+2) \mid n \in \mathbb{Z}\}$. Then clearly F is an open cover of $\mathbb{R}$; but F has no finite subcover, since removing any set $(k,k+2)$ from F causes F to "miss" the point $k+1$.

[8] $F = \{(a,b-1/n) \mid n \in \mathbb{N}\}$. Can you prove F has no finite subcover?
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Example 7. What is wrong with the following argument? $F = \{(-\infty, 5)\} \cup \{(n, \infty) \mid n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}$. $F' = \{(-\infty, 5), (4, \infty)\}$ is a finite subcover of F. So $\mathbb{R}$ is compact. [9]

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Heine-Borel Theorem: Every closed interval $[a, b] \subset \mathbb{R}$ is compact.

The proof is somewhat long and involved, so we skip it here. You can find it in any standard point-set topology textbook (Extra Credit).

Theorem 1. The continuous image of a compact set is compact; i.e., if $f : X \to Y$ is a continuous map between two topological spaces, and if X is compact, then $f(X)$ is compact.

Proof: Homework.

Note. In the above theorem, $f(X)$ is guaranteed to be compact; but Y may or may not be compact.

Corollary 2. Let X and Y be two topological spaces. If $X \simeq Y$, then X is compact iff Y is compact.

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[9] This open cover happens to have a finite subcover, but we didn't prove that *every* open cover of $\mathbb{R}$ has a finite subcover.