

Review

A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X satisfying: $\emptyset$ and X are in $\mathcal{T}$; and $\mathcal{T}$ is closed under unions and finite intersections. X can be any set; elements of X are called **points**. The pair $(X, \mathcal{T})$ is called a **topological space**. $\mathcal{T}$ is a topology. The elements of $\mathcal{T}$ (they are subsets of X) are called **open** sets. A topology on X is a *declaration* of which subsets of X we are *choosing* to call open; we can choose any collection of subsets we desire, as long as the above conditions are satisfied.

Given a topological space $(X, \mathcal{T})$, a subset A of X can be given a topology $\mathcal{T}_A$ by: $V \in \mathcal{T}_A$ iff $V = U \cap A$ for some $U \in \mathcal{T}$. This is called the **subspace topology** (also called the *relative topology*) on A . We say $(A, \mathcal{T}_A)$ is a (topological) **subspace** of $(X, \mathcal{T})$. The topology on A is **induced** by the topology on X .

A function from one topological space to another is **continuous** iff the preimage of every open set is open. A **homeomorphism** (denoted $\simeq$) is a bijection that is continuous and has a continuous inverse.

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### Connectedness

Informal Definition: A **topological invariant** is a property of a topological space that is preserved by homeomorphisms.

*Example 1.* We will soon prove that: If  $A \simeq B$ , then  $A$  is connected iff  $B$  is connected. In other words, “connectedness” is preserved by homeomorphisms, so it is a topological invariant.

Intuitively, we’d like to say the interval  $(0, 3)$  is connected, while  $(0, 1) \cup (2, 3)$  is not.

*Definition 1.* A topological space  $X$  is **connected** iff it is *not* equal to the union of two disjoint nonempty open subsets.

*Example 2.* Let  $X = (0, 1) \cup (2, 3) \subset \mathbb{R}$ , with the subspace topology induced by  $\mathbb{R}$ .

Q: Is each of  $(0, 1)$  and  $(2, 3)$  open in  $X$ ? <sup>1</sup>

Q: Is  $X$  connected? <sup>2</sup>

*Example 3.* Let  $X = [0, 1) \cup (1, 2) \cup [5, 7] \subset \mathbb{R}$ .

Q: Is each of  $[0, 1)$ ,  $(1, 2)$ , and  $[5, 7]$  open in  $X$ ? <sup>3</sup>

Q: Is  $X$  connected? Can you write  $X$  as the union of *two* disjoint open subsets? <sup>4</sup>

*Theorem 1.*  $\mathbb{R}$  is connected.

*Proof.* Suppose towards contradiction that  $\mathbb{R}$  is not connected. Then, by definition,  $\exists A, B \subset \mathbb{R}$  such that  $\mathbb{R} = A \cup B$ , where  $A$  and  $B$  are disjoint nonempty open subsets of  $\mathbb{R}$ . Pick arbitrary points  $a \in A$  and  $b \in B$ . Without loss of generality, we can assume  $a < b$ . Let  $A' = [a, b] \cap A$ . Let  $z = \text{lub}(A')$  ( $A'$  has a least upper bound because every bounded nonempty subset of  $\mathbb{R}$  has a lub — this is called the Completeness Axiom for real numbers). Now, we claim that  $z \notin A$  and  $z \notin B$ . Proof of claim: Extra Credit. But this gives us a contradiction, since  $z \in \mathbb{R} = A \cup B$ .  $\square$

*Theorem 2.*  $A \subseteq \mathbb{R}$  is connected iff  $A$  is an interval (open, closed, or half open; infinite or half-infinite).

<sup>1</sup>Yes. Why? This isn’t as trivial as it seems; prove it using the definition of subspace topology!

<sup>2</sup>No. Why?

<sup>3</sup>Yes. Why?

<sup>4</sup>Let  $U = [0, 1)$ ,  $V = (1, 2) \cup [5, 7]$ ; then  $U$  and  $V$  are disjoint, each is nonempty and open in  $X$ , and  $X = U \cup V$ .

Proof: Extra Credit.

*Theorem 3.* The continuous image of a connected set is connected; i.e, if  $f : X \rightarrow Y$  is a continuous map between topological spaces, and if  $X$  is connected, then  $f(X)$  is connected.

Proof: Homework.

*Note.* In the above theorem,  $f(X)$  is guaranteed to be connected, but  $Y$  may or may not be connected.

*Corollary 4.* If  $X$  is connected, and  $Y$  is homeomorphic to  $X$ , then  $Y$  is connected (i.e., connectedness is a topological invariant).

Proof: Homework.

*Example 4.* Prove  $[a, b] \not\cong (c, d)$ .

Sketch of Proof: Suppose towards contradiction that there exists a homeomorphism  $h : [a, b] \rightarrow (c, d)$ . Let  $X = [a, b] - \{a\}$ ,  $Y = (c, d) - \{h(a)\}$ . It is easy to show that  $Y$  is not connected, but  $X$  is connected (since  $X \simeq \mathbb{R}$ ). It is also easy to show that the restriction  $h|_X : X \rightarrow Y$  is a homeomorphism, which implies that  $Y$  must be connected. This gives us the desired contradiction.

Q: How would you prove that  $[a, b]$  is not homeomorphic to a circle (denoted  $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ )?

Ans: we will prove this rigorously later; here is an informal proof: You need to remove at least two points from  $S^1$  to make it disconnected, but you can disconnect  $[a, b]$  by removing only one point.

*Theorem 5.* A topological space  $X$  is connected iff it contains no proper subset which is both open and closed in  $X$ .

Proof: Homework.

*Theorem 6.* If  $A$  and  $B$  are connected subspaces of a topological space  $X$ , and if  $A \cap B \neq \emptyset$ , then  $A \cup B$  is connected.

Proof: Homework.

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