

The *long line* is an example of a connected, Hausdorff topological space that's locally homeomorphic to \mathbb{R} , but is not a manifold, because it is not second countable (defined below). Roughly speaking, it is $S_\Omega \times [0, 1)$, where S_Ω is the smallest uncountable ordinal. To describe and understand it in detail, we first need some definitions. I am borrowing most of the following from Munkres's book, *Topology*, Second Edition, Prentice Hall.

Definition A relation R on a set X is an **order relation** if $\forall x, y \in X$, (1) xRy or yRx , (2) $\neg(xRx)$, and (3) R is transitive.

Definition A set X is **well ordered** by an order relation $<$ if every nonempty subset of X has a "smallest" element.

We construct the long line as follows. Let S_Ω be the smallest uncountable ordinal (if you don't know what that is, just think of it as a well-ordered uncountable set). Let $L = S_\Omega \times [0, 1) - \{(s_0, 0)\}$, where s_0 is the smallest element of S_Ω . Order L with the dictionary ordering, i.e., $(s, t) < (s', t')$ iff $s < s'$ or $[s = s' \text{ and } t < t']$.

We give L the **order topology**, which means a subset of L is open iff it is a union of open intervals (a, b) with $a, b \in L$, $a < b$. Then L with this topology is called the **long line**.

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#### Extra Credit Problems

1. Explain why we choose  $L = S_\Omega \times [0, 1)$  minus its smallest element instead of just letting  $L = S_\Omega \times (0, 1]$ . Don't read the following hint until you've thought hard. Hint: The second way would not produce a connected space; why?

*Definition* Let  $\mathcal{T}$  be a topology on a set  $X$ . A **basis** for  $\mathcal{T}$  is a subset  $\beta \subseteq \mathcal{T}$  such that  $\forall U \in \mathcal{T}$  and  $\forall x \in U$ ,  $\exists B \in \beta$  with  $x \in B \subseteq U$ .

2. (a) Prove that the long line is not **second countable**, i.e, it does not have a countable basis.  
 (b) Prove that the long line cannot be embedded in  $\mathbb{R}$ , or in any  $\mathbb{R}^n$ . Hint:  $\mathbb{R}^n$  has a countable basis.