## Definitions

*Definition 1.* A **truth assignment** is a function from the propositional variables $\{p_1, p_2, \cdots\}$ to $\{\mathrm{T}, \mathrm{F}\}$. (In other words, we assign a value of T or F to each propositional variable.)

*Definition 2.* A formula $A$ is said to be a **tautology** if for every truth assignment $\phi$, $\phi(A) = \mathrm{T}$.

*Definition 3.* A formula $A$ is said to be a **satisfiable** if for some truth assignment $\phi$, $\phi(A) = \mathrm{T}$; we say $\phi$ **satisfies** $A$, or $A$ is **satisfied** by $\phi$.

*Definition 4.* Let $S$ be a set of formulas. A truth assignment $\phi$ **satisfies** $S$ if for every $A \in S$, $\phi(A) = \mathrm{T}$. $S$ is said to be a **satisfiable** if there exists a truth assignment that satisfies $S$.

*Definition 5.* Let $B$ be a formula, and $S$ a set of formulas. We say $B$ is a **tautological consequence** of $S$ if every truth assignment that satisfies $S$ also satisfies $B$; we write $S \models B$.

*Definition 6.* Two formulas $A$ and $B$ are said to be **logically equivalent** iff the formula $(A \leftrightarrow B)$ is a tautology.

*Definition 7.* An $n$**-ary truth function** is a function from $\{\mathrm{T}, \mathrm{F}\}^n \to \{\mathrm{T}, \mathrm{F}\}$.

*Definition 8.* A given set of connectives is said to be **adequate** iff for every truth function $G : \{\mathrm{T}, \mathrm{F}\}^n \to \{\mathrm{T}, \mathrm{F}\}$, there exists a formula $A$ that uses only the given connectives, such that $H_A = G$.

*Definition 9.* We agree on the following abbreviations: **1**. $\neg(\neg A \vee \neg B)$ is abbreviated by $A \wedge B$. **2**. $(\neg A \vee B)$ is abbreviated by $A \to B$. **3**. $(A \to B) \wedge (B \to A)$ is abbreviated by $A \leftrightarrow B$.

*Definition 10.* Let $S$ be a set of formulas. To give a **proof** of a formula $B$ **using** the formulas in $S$ means to write a list of formulas such that: 1. Each formula in the list is either an axiom or is in $S$ or is obtained from the previous formulas in the list by rules of inference. 2. The last formula in the list is $B$. We write: $S \vdash B$ (read: $S$ proves $B$). If $S$ happens to be the empty set, i.e., we are not using any formulas as hypothesis, then we write $\vdash B$, which says that $B$ can be proved from just the axioms and rules of inference, without using any hypotheses. For emphasis, we sometimes write $\vdash_{\mathrm{P}}$ instead of $\vdash$.

*Definition 11.* Let $S$ be a set of formulas. If $\exists B$ such that $S \vdash B$ and $S \vdash \neg B$, then we say $S$ is **inconsistent**. If there is no such $B$, then we say $S$ is **consistent**.

*Definition 12.* A formal system is said to be **decidable** if there is an algorithm (a systematic method) for deciding whether any given formula has a proof or not.

*Definition 13.* We say that two sets $S$ and $T$ are **equipotent** or have the **same size**, written as $|S| = |T|$ or $S \approx T$, iff there exists a bijection $f : S \to T$.

*Definition 14.* Any set that has the same size as $\mathbb{N}$ is said to be **denumerable**. A set is called **countable** if it's either finite or denumerable. A set is called **uncountable** if it is not countable.

*Definition 15.* Suppose $S$ and $T$ are sets, and $f : S \to T$ a map. If $f$ is 1-1, we write $|S| \leq |T|$. If $f$ is onto, we write $|S| \geq |T|$.

*Definition 16.* For any formula $A$, "$\exists x_i A$" stands for "$\neg \forall x_i \neg A$."

*Definition 17.* A formula in a FOL L is said to be **logically valid (LV)** if it is true in every interpretation of L.

*Definition 18.* Let $A$ and $B$ be two formulas in some FOL L. We say $A$ and $B$ are **logically equivalent (LE)** if the formula $A \leftrightarrow B$ is logically valid.

*Definition 19.* (Not in our book) Suppose $A$ is a formula that contains free variables. Then $A$ is **true** in an interpretation iff its closure is true in that interpretation.

*Definition 20.* (Informal) Suppose $A$ is a formula that contains free variables. The **closure** of $A$ is obtained by quantifying every free variable of $A$ with a $\forall$.

*Definition 21.* Suppose $A$ has free variables. $A$ is **false** in an interpretation iff $\neg A$ is true in that interpretation.

*Definition 22.* (Informal) A term $t$ is said to be **substitutable** for $x$ in $A$ if no variable in $t$ becomes bound after the substitution.

*Definition 23.* (Differs from book) If $A$ is true in every model of $\Gamma$, then we denote this by $\Gamma \models A$. If $\Gamma = \phi$, then we write $\models A$, which means $A$ is true in every interpretation of L.

*Definition 24.* A set $\Gamma$ of formulas in L is **consistent** iff there is no formula $A$ such that $\Gamma$ proves both $A$ and $\neg A$.

*Definition 25.* The argument form $A_1, \cdots, A_n \therefore B$ is said to be **valid** iff $\{A_1, \cdots, A_n\} \models B$.

*Definition 26.* (Informal) A relation is said to be **decidable** if there is an algorithm for deciding whether the relation is true or false for any given input; i.e., given any input, the algorithm will stop after finitely many steps and give an output of YES or NO (or T or F).

*Definition 27.* (Informal) A relation $R$ is **semidecidable** if there is an algorithm that, given any input $x$, stops with output YES if $R(x) = \mathrm{T}$, and doesn't stop if $R(x) = \mathrm{F}$.

## Theorems

*Theorem 1.* $(A_1 \wedge \cdots \wedge A_n) \rightarrow B$ is a tautology iff $\{A_1, \cdots, A_n\} \models B$

*Theorem 2.* (Replacement Theorem) Suppose $A$ and $B$ are logically equivalent formulas, and $C$ is a formula in which $A$ appears. Then, if we replace $A$ with $B$ in $C$, we obtain a formula that is logically equivalent to $C$.

*Theorem 3.* (Adequacy Theorem for connectives) The set of connectives $\{\neg, \vee\}$ is adequate. The set of connectives $\{\neg, \wedge\}$ is also adequate.

*Theorem 4.* (The Soundness Theorem for P: Special case) Every theorem of Propositional Logic is a tautology; i.e., for every formula $A$, if $\vdash A$, then $\models A$.

*Theorem 5.* (The Soundness Theorem for P: General case) Let $S$ be any set of formulas. Then every theorem of $S$ is a tautological consequence of $S$; i.e., for every formula $A$, if $S \vdash A$, then $S \models A$.

*Theorem 6.* (The Adequacy Theorem for P: Special Case) Every tautology is a theorem of Propositional Logic; i.e., for every formula $A$, if $\models A$, then $\vdash A$.

*Theorem 7.* (The Adequacy Theorem for P: General Case) Let $S$ be any set of formulas. If $S \models A$, then $S \vdash A$.

*Theorem 8.* If $|S| = |T|$ and $T$ is denumerable, then $S$ is denumerable.

*Theorem 9.* If $S \subset T$ and $T$ is countable, then $S$ is countable.

*Theorem 10.* If $f : S \rightarrow T$ is onto and $S$ is countable, then $T$ is countable.

*Theorem 11.* If $f : S \rightarrow T$ is 1-1 and $T$ is countable, then $S$ is countable.

*Theorem 12.* (George Cantor) $\mathbb{R}$ is uncountable.

*Theorem 13.* (Compactness Theorem, Version I) Let $\Gamma$ be an infinite set of formulas. If every finite subset of $\Gamma$ is satisfiable, then $\Gamma$ is satisfiable.

*Theorem 14.* (Compactness Theorem, Version II) Let $\Gamma$ be an infinite set of formulas, $A$ any formula. If $\Gamma \models A$, then $\Gamma$ has a finite subset $\Delta$ such that $\Delta \models A$.

*Theorem 15.* If $A \vee B$ is true in an interpretation $I$, and if $x$ does not occur free in $A$, then $A \vee \forall x B$ is true in $I$.

*Theorem 16.* (Soundness Theorem for first order logic) If $\Gamma \vdash A$, then $\Gamma \models A$.

*Theorem 17.* First order logic is consistent; i.e., for any first order language L, there is no formula $A$ in L such that $\vdash A$ and $\vdash \neg A$.

*Theorem 18.* $\Gamma$ has a model iff it's consistent.

*Theorem 19.* The argument form $A_1, \cdots, A_n \therefore B$ is valid iff $\{A_1, \cdots, A_n\} \vdash B$.