Review defs: Tautological consequence. Logically valid.

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*Note.* Throughout this section, unless stated otherwise, L denotes a first order language, $A$ a formula in L, and $\Gamma$ a set of formulas in L.

*Definition 1.* (Differs from book) If $A$ is true in every model of $\Gamma$, then we denote this by $\Gamma \models A$. If $\Gamma = \phi$, then we write $\models A$, which means $A$ is true in every interpretation of L.

*Theorem 1.* (Soundness Theorem for first order logic) If $\Gamma \vdash A$, then $\Gamma \models A$.

*Proof.* (Sketch) Suppose $\Gamma \vdash A$. WTS $\Gamma \models A$, which means $A$ is true in every model of $\Gamma$.

Let $I$ be a model of $\Gamma$. WTS $A$ is true in $I$. Step 1: All the axioms are true in $I$. Step 2: Every formula in $\Gamma$ is true in $I$ b/c $I$ is a model of $\Gamma$, and by def of model. Step 3: If a rule of inference is used to derive a formula $B$ from other formulas that are true in $I$, then $B$ is true in $I$. □

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*Definition 2.* A set $\Gamma$ of formulas in L is **consistent** iff there is no formula $A$ such that $\Gamma$ proves both $A$ and $\neg A$.

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*Example 1.* Q: Let $\Gamma = \{\exists x(x < S0 \wedge S0 < x)\}$. Is this set consistent? Ans: Yes! The formula in $\Gamma$ is false in the standard interp of $\mathrm{L_{NN}}$; but $\Gamma$ is not inconsistent: there is no formula $A$ such that $\Gamma$ proves both $A$ and $\neg A$. How do we prove this? By proving that it has a model, as we'll see shortly.

Q: Let $\Gamma = \{\exists x(x < S0 \wedge S0 < x), \forall x\forall y(x = y \vee x < y \vee y < x)\}$. Is this set consistent? Ans: Still yes!

Q: Let $\Gamma = \{\exists x(x < S0 \wedge S0 < x), \forall x\forall y(x = y \vee x < y \vee y < x), \neg\exists x\exists y(x < y \wedge y < x)\}$. Is this set consistent? Ans: No. How can we prove the answer is no? Find a formula $A$ such that $\Gamma \vdash A$ and $\Gamma \vdash \neg A$. One possibility for $A$ is: $\exists x(x < S0 \wedge S0 < x)$. Then clearly $\Gamma \vdash A$; so now we only need to show $\Gamma \vdash \neg A$, as follows: 1. $\neg\exists x\exists y(x < y \wedge y < x)$ HYP. 2. $\neg\neg\forall x\neg\exists y(x < y \wedge y < x)$ DEF $\exists$. 3. $\forall x\neg\exists y(x < y \wedge y < x)$ $\neg\neg$ Rule. 4. $\neg\exists y(x < y \wedge y < x)$ $\forall$ ELIM. 5. $\neg\neg\forall y\neg(x < y \wedge y < x)$ DEF $\exists$. 6. $\forall y\neg(x < y \wedge y < x)$ $\neg\neg$ Rule. 7. $\neg(x < S0 \wedge S0 < x)$ SUBST RULE. 8. $\forall x\neg(x < S0 \wedge S0 < x)$ GEN. 9. $\neg\neg\forall x\neg(x < S0 \wedge S0 < x)$ $\neg\neg$ Rule. 10. $\neg\exists x(x < S0 \wedge S0 < x)$ DEF $\exists$.

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*Theorem 2.* First order logic is consistent; i.e., for any first order language L, there is no formula $A$ in L such that $\vdash A$ and $\vdash \neg A$.

*Proof.* Suppose, towards contradiction, that there is an $A$ such that $\vdash A$ and $\vdash \neg A$. Then, by the Soundness Theorem, $\models A$ and $\models \neg A$. This means in every interpretation $I$ of L, both $A$ and $\neg A$ are true, which is impossible. □

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Q: T or F? If $\Gamma$ has a model, then it's consistent. Ans: See the following theorem.

*Theorem 3.* $\Gamma$ has a model iff it's consistent. Proof: Omitted.

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*Example 2.* Prove that the set $\Gamma = \{\exists x(x < S0 \wedge S0 < x)\}$ is consistent.

Ans: We will show $\Gamma$ has a model. Then, by the above thm, it is consistent.

Let $I$ be the following interpretation: Interpret everything as in the standard model of $\mathrm{L_{NN}}$, except for $<$, which is interpreted as "divides": $x < y$ means $x|y$.

Then the formula $\exists x(x < S0 \wedge S0 < x)$ is true in $I$, b/c 1 divides itself, so we can let $x$ be 1.

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*Example 3.* [Page 181, problem 4(6)] Is the following argument valid? Anyone who likes Sarah likes Jennifer. Alice does not like Jennifer. $\therefore$ Alice does not like Sarah.

Common sense tell us the answer is *yes*. But can we *prove* it somehow?

We define our language to have one binary relation symbol, $R$, and three constant symbols, $a, j, s$.

Then we write: $\forall x[R(x,s) \to R(x,j)]$, $\neg R(a,j) \therefore \neg R(a,s)$.

Let $\Gamma = \{\forall x[R(x,s) \to R(x,j)], \neg R(a,j)\}$. Then we ask whether $\Gamma \vdash \neg R(a,s)$. (We expect the answer to be yes. Can you verify it?)

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*Definition 3.* The argument form $A_1, \cdots, A_n \therefore B$ is said to be **valid** iff $\{A_1, \cdots, A_n\} \models B$.

*Theorem 4.* The argument form $A_1, \cdots, A_n \therefore B$ is valid iff $\{A_1, \cdots, A_n\} \vdash B$.

*Proof.* The "if direction" follows immediately from the Soundness Theorem. The "only if direction" follows immediately from the Adequacy Theorem (which we'll talk more about later).

$\square$

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**HW #20, due Mon 23 Apr**: Read sec 5.2. Do: p. 181: 7, 11, 12, 14(1), 15(1).