Truth for non-closed formulas

Example 1. Q: True or false: if x is even, then $3x$ is even. Ans: T.

Q: Write the above in $\mathrm{L_{NN}}$. Ans: $[\exists n(x = SS0 \times n)] \to [\exists n(SSS0 \times x = SS0 \times n)]$.

Q: Is the above formula closed (i.e., a sentence)? Ans: No; x is free in it.

When we read the English version "if x is even, then $3x$ is even", we automatically know that it's implicitly saying *for all* x, even though it doesn't say it explicitly.

Conclusion: From now on, we agree on the following convention:

Definition 1. (Not in our book) Suppose A is a formula that contains free variables. Then A is **true** in an interpretation iff its closure is true in that interpretation.

Definition 2. (Informal) Suppose A is a formula that contains free variables. The **closure** of A is obtained by quantifying every free variable of A with a $\forall$.

Example 2. Let A be the formula: $[\exists x(y = SS0 \times x)] \to [\exists z(y + S0 = SS0 \times z)]$.

Q: Which variables are free in A?

Q: Is A true in the standard interpretation of $\mathrm{L_{NN}}$?

Ans: Only y is free. So we ask ourselves: Is $\forall y A$ true? The answer is No.

Example 3. Let A be the formula: $[y = SS0 \times x] \to [\exists z(y + S0 = SS0 \times z)]$.

Q: Which variables are free in A?

Q: Is A true in the standard interpretation of $\mathrm{L_{NN}}$?

Ans: x and y are free. So we ask ourselves: Is $\forall x \forall y A$ true? The answer is Yes! (False implies False.)

For non-closed formulas, not-true $\neq$ false!

Definition 3. Suppose A has free variables. A is **false** in an interpretation iff $\neg A$ is true in that interpretation.

This definition, which seems natural and "innocent", has a surprising consequence: not-true $\neq$ false!

Example 4. Suppose A contains x as a free var.

Q: What does it mean to say A isn't true? Ans: It means $\forall x A$ isn't true.

Q: What does it mean to say A is false? Ans: It means $\forall x(\neg A)$ is true.

These are not the same! Why?

Q: Give an example for which A is neither true nor false. Ans: $x < S0$.
