Example 1. Let A be the following formula in Lnn: $\exists x \forall y (y < x)$.

Q: Is A true in the *standard interpretation* of Lnn? (Domain = $\mathbb{N}$, etc.)

Ans: No, there is no x that's larger than every natural number.

Q: Find an interpretation in which A is true. Ans: Domain = $\{0, 1, 2\}$. Then, if $x = 3$, for all y, $y < x$.

But how do we *define* when a formula is true or false in a given interpretation? Intuitively we know what "true" means in any given interp, but "intuitively" is not enough!

Q: How did we define truth in Prop Logic? (I don't mean a tautology.) Ans: Pick truth values for every prop var; then use the truth tables we've defined for $\vee$ and $\neg$ to compute truth val for A "from inside out."

Example 2. Let A be: $(p \vee q) \rightarrow q$. Then ...

Our book gives a precise def for truth in FOLs. But we'll just rely on our intuition. Read the book if interested. We'll just cover a few topics from this section.

Definition 1. A formula in a FOL L is said to be **logically valid (LV)** if it is true in every interpretation of L.

Example 3. Let L be a FOL with a binary relation R. Determine whether each of the following is LV.

(a) A: $\exists y \forall x [R(x, y)] \rightarrow \forall x \exists y [R(x, y)]$.

Ans: True. We can't prove this rigorously without knowing the precise definition of truth. So we'll only explain our reasoning: Antecedent: There is a y such that every x is related to it; let's denote this y by k. So the conclusion is true: For every x, there is a y, namely k, such that x is related to it.

(b) B: $\forall x \exists y [R(x, y)] \rightarrow \exists y \forall x [R(x, y)]$.

Ans: B is not LV. How do you prove this? Find an interp which makes B false. Let I be: domain: $\mathbb{N}$; $R(x, y)$: y is twice x. Then the Antecedent of B, $\forall x \exists y [R(x, y)]$, is true (for every number there is a number that's twice it); but it's conclusion, $\exists y \forall x [R(x, y)]$ is false (there is no number that's twice every number).

Definition 2. Let A and B be two formulas in some FOL L. We say A and B are **logically equivalent (LE)** if the formula $A \leftrightarrow B$ is logically valid.

Example 4. Suppose C and D are two given formulas in some FOL L. Let A be the formula $\forall x (C \rightarrow D)$; Let B be the formula $(\forall x C) \rightarrow (\forall x D)$. Are A and B LE, no matter what C and D are?

Ans: No! $A \leftrightarrow B$ is not LV. To prove this, we'll find an interp and specific formulas C and D such that B is true and A is false.

I: domain: $\mathbb{N}$. Let C be the formula $\exists y (x = y + y)$; let D be the formula $\exists y (y < 0)$.

D is false; but C is true when x is even. So A is false; why?

Now, $\forall x C$ is false; why? So B is true! (Why?) So A and B are not LE.

Q: $A \leftrightarrow B$ is not LV; so either $A \rightarrow B$ is not LV, or $B \rightarrow A$ is not LV, or both. Which is the case? Ans: $A \rightarrow B$ is LV, but $B \rightarrow A$ is not LV; why?

HW #13, due Wed 28 Mar: Read Section 4.2. Do: p. 141: 27, 28(1,2), 29, 32(2abc).

HW #14, due Fri 30 Mar: Read Parts of section 4.3 done in class. Do: p. 152: 13(2,4).