Consider the argument: if it snows, it's cold; it snows; therefore it's cold. We can represent this symbolically as: $[(s \to c) \wedge s] \to c$ (where $s =$ it snows, $c =$ it's cold).

Now try representing the following symbolically: all dogs are animals; there exists a dog; therefore there exists an animal. Without symbols representing "all" and "exists", we don't have a good way to do this. Using the symbols $\forall$, $\exists$, we can write: $[\forall x(D(x) \to A(x)) \wedge \exists x D(x)] \to \exists x A(x)$ (where, as you can guess, $D(x) =$ "x is a dog"; $A(x) =$ "x is an animal").

The symbols $\forall$ and $\exists$ are called **quantifiers**. When we add these quantifiers to Propositional Logic, we call it **First-Order Logic**—also called *Predicate Logic* or *Predicate Calculus*, as well as other names!

What does "First-Order" mean? Roughly speaking, it means we quantify only at the first "level", over variables, not at any higher "levels", such as over formulas, etc. This will make more sense later. (We do quantify over formulas, or even higher levels, such as over proofs, in the meta-language, i.e., when we talk *about* Propositional Logic or First-Order Logic, but that's not *in* the language itself.)

Example 1. Write each of the following using mathematical symbols.

1. x is prime. Ans: $\forall n[(n|x) \to (n = 1 \vee n = x)]$. ($a|b$ means a divides b; more about this below.)
2. x and y are twin primes.
   Ans: $\forall n[(n|x) \to (n = 1 \vee n = x)] \wedge \forall n[(n|y) \to (n = 1 \vee n = y)] \wedge [x + 2 = y]$.
3. There exist arbitrarily large twin primes. (This is an open conjecture!)
   Ans:
   $\forall z(\exists x \exists y[x > z] \wedge [y > z] \wedge \forall n[(n|x) \to (n = 1 \vee n = x)] \wedge \forall n[(n|y) \to (n = 1 \vee n = y)] \wedge [x+2 = y])$.

Propositional Logic:	Semantics: (Chapter 2) Truth values	Syntax: (Chapter 3) Formal systems, Proofs
First-Order Logic (FOL):	Semantics: (Chapter 4) Truth values Models, Interpretation	Syntax: (Chapter 5) Formal Systems, Proofs

There are different First-Order Languages: FOL for Arithmetic, FOL for Set Theory, FOL for Group Theory, etc. They all share the same logical symbols: $\wedge, \vee, \neg, \to, \leftrightarrow, \exists, \forall, =, (,)$. But each language has its own extra symbols; e.g., Arithmetic has $<, +, \times$, etc.; Set Theory has: $\in, \subset$, etc.

Lnn: The language of Arithmetic (nn= natural numbers)

1. Symbols
   (a) Logical Symbols: $\wedge\ \vee\ \neg\ \to \leftrightarrow\ \exists\ \forall\ =\ (\ )\ x_1\ x_2\ x_3 \cdots$.
   (b) Nonlogical (or language-specific) Symbols: $0\ S\ +\ \times\ <$.
   (S is the successor symbol: $S0$ means 1, $SS0$ means 2, so on. Purpose: to avoid including the symbols $1, 2, 3, \cdots$ in our language. Why? To keep it "small", for coding purposes for Gödel coding.)
2. Terms and Formulas: Read in the book!

   Quick examples:
   Term: $(x_1 + x_2) \times (SS0 + x_1)$
   Formula: $(x_1 < x_2) \wedge \neg(SS0 + S0 = x_1)$

As in Prop Logic, we allow ourselves some conveniences: use letters other than just x_i; use [ ] instead of ( ); drop parenthesis when it doesn't result in ambiguity; etc.

Example 2. Write each of the following in Lnn.

1. x is prime.

   Note: Cannot write $n|x$; the symbol | is not in Lnn. So what can we do?

   Ans: Replace $n|x$ by $\exists m(m \times n = x)$. So we get: $\forall n[(\exists m(m \times n = x)) \rightarrow (n = 1 \vee n = x)]$.

2. x and y are twin primes.

   Note: Cannot write $x + 2 = y$; the symbol 2 is not in Lnn. So what can we do?

   Ans: $\forall n[(\exists m(m \times n = x) \rightarrow (n = 1 \vee n = x)] \quad \wedge \quad \forall n[(\exists m(m \times n = y) \rightarrow (n = 1 \vee n = y)] \wedge [x + S00 = y]$.

Example 3. Translate into English:

1. $\forall x_1 \exists x_2 (SS0 \times x_1 = x_2)$. Ans: For every number, there is a number twice it (true).
2. $\exists x_2 \forall x_1 (SS0 \times x_1 = x_2)$. Ans: There is a number that's twice every number (false).

Scope; bound and free variables; sentences

Example 4.

1. Is this formula true: $\exists m \exists n(m + n = SSS0)$? (We haven't defined truth yet; we're speaking informally.) Ans: Yes. One possibility (among others): $m = S0$, $n = SS0$.
2. Is this formula true: $\exists m \forall n(m + n = SSS0)$? Ans: No.
3. Is this formula true: $\exists m(m + n = SSS0)$? Ans: The question is meaningless: What's n?

   • In (1) and (2) above, m and n are **bound** variables; in (3), m is bound, but n is **free**. A formula in which every variable is bound is called a **sentence**, or a **closed** formula. So (1) and (2) are sentences, but (3) isn't.

4. Is this formula true: $\exists m \exists n[(m + n = SSS0) \wedge (SSS0 < m)]$? Ans: No, there is no such m and n.
5. Is this formula true: $[\exists m \exists n(m + n = SSS0)] \wedge [SSS0 < m]$? Ans: The question is meaningless: What's the last m?

   • In this last formula, the last m is not within the **scope** of the $\exists m$; so the first two m's are bound, the last m is free.

HW # 10, due Wed 07 Mar
Read Section 4.1. Do: p. 134: 2(1,6,7), 3(1,2,4,5,7-9).

HW # 11, due Fri 09 Mar
Read Section 4.1. Do: p. 134: 4, 5, 6(2,4), 7.