Review: What does it mean for a set Γ of formulas to be satisfiable?

Theorem 1. (Compactness Theorem) Let Γ be an infinite set of formulas. If every finite subset of Γ is satisfiable, then Γ is satisfiable.

Proof: Skip (may do later).

Equivalent Statement (Contrapositive): If an infinite set of formulas is not satisfiable, then some finite subset of it is not satisfiable!

To appreciate the Compactness Theorem better, consider the following examples.

Example 1. An infinite set of inequalities. Let $S = \{0 < x < 1,\ 0 < x < 1/2,\ 0 < x < 1/3,\ 0 < x < 1/4,\ \cdots\}$.

Q: Does S have any solutions? Ans: No; for x to satisfy all equations, it has to be less than $1/n$ for every n; but for any given positive number x, there is a large enough n such that $1/n < x$.

Q: Does every finite subset of S have a solution? Ans: Yes. Why?

FM equations

We'll use the letters x, y, z for variables (or $x_1, x_2, \cdots$, when we need more than three). We work $\bmod 2$, i.e, only with 0 and 1.

Let's consider equations that use only the following two operations:

1. Multiplication. Examples: xy, xxz, etc.

2. Taking f of an expression, where f denotes the function $f(x) = 1 - x$. Examples: $f(y)$, $f(xy)$, $f(xf(z))$, etc.

(FM stands for the function f and multiplication.)

Example 2. Solve each of the following FM equations.

1. $f(x)y = 0$. Ans: $x = 1$ or $y = 0$.
2. $f(f(x)y) = 0$. Ans: $x = 0$ and $y = 1$.
3. $xf(x) = 0$. Ans: $x = 0$ or $x = 1$.

Example 3. Solve the set of FM equations $S = \{f(x)y = 0, xf(y) = 0\}$. Ans: $(x, y) = (0, 0)$, or $(x, y) = (1, 1)$.

Example 4. Solve the set of FM equations $S = \{x_1 f(x_2) = 0, x_2 f(x_3) = 0, x_3 f(x_4) = 0, \cdots\}$. Ans: $\forall i$, $x_i = 0$, or $\forall i$, $x_i = 1$.

Theorem. Let S be an infinite set of FM equations. If every finite subset of S has a solution, then S has a solution.

Proof: Skip (same as the Compactness Theorem).

The Book

There is a book of names, called The Book. To write our name in The Book, we must follow these rules:

1. (AXIOMS) For any name-combo X, may write <u>X</u> X.
2. (EXP) For any name-combo X, we may write Y X, if Y is a line that's already in The Book.
3. (CONTRACTION) If some line in The Book is of the form X X, then may write X.

4. (CUT) If The Book contains lines of the form X Y and $\underline{\text{X}}$ Z, then may write Y Z.

What's a name-combo?

Def: A name-combo is any finite sequence of (proper) names written next to each other, with underlines appearing anywhere in any combination.

Example 5. How can we write the name-combo Bob Alice $\underline{\text{Bob}}$ in The Book?

Ans:

1. $\underline{\text{Bob}}$ Bob (AXIOM)
2. $\underline{\text{Bob}}$ Bob Alice (EXP)
3. $\underline{\underline{\text{Bob}}}$ $\underline{\text{Bob}}$ (AXIOM)
4. Bob Alice $\underline{\text{Bob}}$ (CUT)

Some name-combos cannot be written in The Book, no matter how hard we try! Two examples: 1. Bob. 2. $\underline{\text{Bob}\ \underline{\text{Bob}}}$. We'll prove this shortly.

Q: Is there an algorithm for deciding whether a given name-combo can or cannot be written in The Book?

Ans: Yes:

<ins>Step 1</ins> To each name-combo we associate a FM expression by: Replace each name by a variable (distinct variables for distinct names). For each underline write the function f.

Example 6. "Bob Alice $\underline{\text{Bob}}$" becomes: $xyf(x)$.

<ins>Step 2</ins> Use the following theorem.

Theorem 2. (Soundness + Adequacy) A name-combo can be written in The Book iff its associated FM expression is always 0.

Example 7. "Bob Alice $\underline{\text{Bob}}$" becomes: $xyf(x)$, which is always 0 because either x or $f(x)$ is always 0, so by Adequacy, "Bob Alice $\underline{\text{Bob}}$" can be written in The Book.

"Bob" becomes x, which can be both 0 and 1; so by Soundness, it cannot be written in The Book.

$\underline{\text{Bob}\ \underline{\text{Bob}}}$ is always 1, never 0 (why?); so by Soundness, it cannot be written in The Book.

As you probably guessed by now:

FM Expressions: 0 corresponds to T, 1 to F. The function f corresponds to $\neg$. Multiplication corresponds to $\vee$. "A FM expression is always 0" corresponds to "tautology".

The Book: Underline corresponds to $\neg$. Writing name-combos next to each other corresponds to $\vee$. A name-combo that can be written in The Book corresponds to a formula that has a proof in Prop Logic.

The point of all this: A priori, FM expressions have nothing to do with The Book. Similarly, a priori, truth has nothing to do with proof. The Adequacy and Soundness Theorems are what relate these independent notions to each other.

HW # 9, due Fri 02 Mar
Reread Section 2.2. Do: p. 64: 21, 26, 27, 35, 37.