The method of diagonalization is a common and very powerful tool used in a variety of fields in mathematics. George Cantor originally invented it to prove, among other things, that $\mathbb{R}$ is not countable.

Theorem 1. (George Cantor) $\mathbb{R}$ is uncountable.

Proof. Let f be a function from $\mathbb{N}$ to $\mathbb{R}$. We will show that f is not onto. Therefore there is no bijection from $\mathbb{N}$ to $\mathbb{R}$, so $\mathbb{R}$ is uncountable.

For each $n \in \mathbb{N}$, write the real number

Define a real number $x \in [0, 1]$ as follows:

$$\text{the } n\text{th digit of } x \text{ after the decimal point} = \begin{cases} 1 & \text{if the } n\text{th digit of } f(n) \text{ after the decimal } = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then, for every $n \in \mathbb{N}$, $x \neq f(n)$, since they have different nth digits, by construction of x. So $\nexists n \in \mathbb{N}$ such that $f(n) = x$; i.e., f is not onto. □

Recall: $|S| \leq |T|$ means: (1) there is a 1-1 function from S to T; or (2) there is an onto function from T to S. (It can be shown that these two conditions are equivalent.)

Q: What would you guess $|S| < |T|$ means? Ans: It means $|S| \leq |T|$ but $|S| \neq |T|$.

It follows from the above theorem that $|\mathbb{N}| < |\mathbb{R}|$. Now, a natural questions arises:

Q: Is there a set S such that $|\mathbb{N}| < |S| < |\mathbb{R}|$?

The Continuum Hypothesis (CH) There is no set S such that $|\mathbb{N}| < |S| < |\mathbb{R}|$.

Gödel (1938): CH is consistent with Set Theory.

Cohen (1963): The negation of CH is consistent with Set Theory.

In other words, the axioms of Set Theory can neither prove nor disprove CH!

HW # 8, due Wed 28 Feb
Read Section 1.5. Do: p. 25: 1(1,3), 5, 15, 16.