Q: How would you check if two boxes of apples have the same number of apples in them, if you didn't know how to count? Ans: By pairing them off.

"Pairing off" is equivalent to finding a bijection between the two sets of apples. This notion applies to infinite sets as well as finite ones. We'll see that some infinite sets are larger than others!.

Review: Let $f : S \to T$ be a function from one set to another. f is **one-to-one** iff different inputs always produce different outputs, i.e., if $x \neq y$, then $f(x) \neq f(y)$. f is **onto** iff every element in T "gets hit", i.e., if $\forall b \in T$, $\exists a \in S$ such that $f(a) = b$; or equivalently, if $f(S) = T$ (instead of just $f(S) \subset T$). f is a **bijection** iff it's both 1-1 and onto.

Synonyms: **injective** = 1-1; **surjective** = onto.

Definition 1. We say that two sets S and T are **equipotent** or have the **same size**, written as $|S| = |T|$ or $S \approx T$, iff there exists a bijection $f : S \to T$.

Other Synonyms for "have the same size": "have the same cardinality", or "are bijective".

Example 1. (a) $S = \{1, 2, 3\}$, $T = \{L, 5, \pi\}$ are equipotent. Bijection=?

(b) The set of positive odd numbers has the same size (cardinality) as the set of positive even numbers. Bijection=?

(c) (A little surprising) The set of positive even numbers has the same size as the set of all natural numbers. Bijection=?

(d) (A little more surprising) Fact: between any two numbers there are infinitely many rational numbers. Nevertheless, the set of all rational numbers has the same size as the set of all natural numbers! (Proof in HW.)

Not all infinite sets have the same size. For example, $\mathbb{R}$ does not have the same size as $\mathbb{N}$, as we'll see later.

Definition 2. Any set that has the same size as $\mathbb{N}$ is said to be **denumerable**. A set is called **countable** if it's either finite or denumerable. A set is called **uncountable** if it is not countable.

Example 2. Is each of the sets in the above example countable or uncountable? Ans: They're all countable. Which ones are denumerable? The infinite ones.

Example 3. Show that $\mathbb{Z}$ is countable.

Proof: We give an *enumeration* of $\mathbb{Z}$: 0, 1, -1, 2, -2, 3, -3, 4, -4, ...

This implicitly defines a bijection $f : \mathbb{N} \to \mathbb{Z}$ (or vice versa). What is f?

<u>Informal Definition:</u> An **enumeration** of a set S is a list of its elements, $a_1, a_2, a_3, \cdots$, such that every element appears at least and at most once, and there is a clear pattern how the list continues.

To prove a set is countable, it is sufficient, and sometimes easier, to enumerate it. Why? Because we just need to see a clear pattern, instead of coming up with a precise formula.

Example 4. Let S be the set of all words using the two letters a and b. (For instance, a, and $babbbaa$ are two such words.)

Q: Is S finite or infinite? Ans: Infinite; S contains the subset $\{a, aa, aaa, \cdots\}$, which is clearly an infinite set.

Q: Is S countable? Ans: Yes. We prove this by enumerating S:

$a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, bab, bba, bbb, aaaa, \cdots$.

Theorem 1. If $|S| = |T|$ and T is denumerable, then S is denumerable.

Proof. $|S| = |T|$, so by def, $\exists$ bijection $f : S \to T$. T is denumerable, so by def, $\exists$ bijection $g : T \to \mathbb{N}$. Then $g \circ f$ is a bijection from S to $\mathbb{N}$. $\square$

Comparing Size

Example 5. Suppose S and T are finite sets, and $f : S \to T$ a map.

Q: If f is 1-1, what can we say about the sizes of the two sets? Ans: T has at least as many elements as S.

Q: If f is onto, what can we say about the sizes of the two sets? Ans: S has at least as many elements as T.

This idea "works" for infinite sets too:

Definition 3. Suppose S and T are sets, and $f : S \to T$ a map. If f is 1-1, we write $|S| \leq |T|$. If f is onto, we write $|S| \geq |T|$.

Theorem 2. If $S \subset T$ and T is countable, then S is countable.

Proof: Skip (not easy).

Corollary 3. If $f : S \to T$ is 1-1, and T is countable, then S is countable.

Proof. $f(S) \subset T$, so by the above theorem, $f(S)$ is countable. f is a bijection from S to $f(S)$ (why?). So, by the above lemma, S is countable. $\square$

Theorem 4. If $f : S \to T$ is onto, and S is countable, then T is countable.

Proof: Skip (see p. 25, problem 9, if interested).

Informal Definition: An **exhaustive list** of a set S is a list of its elements, $a_1, a_2, a_3, \cdots$, such that every element appears at least once, and there is a clear pattern how the list continues.

To prove a set is countable, exhaustive lists are sufficient (why?), and sometimes easier than enumerations.

Example 6. Use an exhaustive list to prove $\mathbb{Q}$ is countable.

HW # 8, due Mon 26 Feb

Read Section 1.5. Do: p. 25: 1(1,3), 5, 15.