This section is mainly about two questions:

Q1: Is every tautology a theorem?

Q2: Is every theorem a tautology?

What do these questions mean? Review definitions, and discuss.

We'll see that the answer to both questions is: Yes!

Soundness

Theorem 1. (The Soundness Theorem: Special case) Every theorem of Propositional Logic is a tautology; i.e., for every formula A, if $\vdash A$, then $\models A$.

Sketch of proof: Q: Is every axiom a tautology? Yes. Why?

Q: For each of the four rules of inference, check: if the antecedent is a tautology, does it follow that the conclusion is a tautology?

Q: How does this prove the theorem? A: Suppose we have a proof, i.e., a list of formulas: $A_1, \cdots, A_n$. A_1 is guaranteed to be a tautology. Why? So A_2 is guaranteed to be a tautology. Why? So A_3 is guaranteed to be a tautology. Why? And so on.

Q: What is a more rigorous way to prove this? Ans: Use induction.

Review: What does $S \vdash A$ mean? What does $S \models A$ mean?

Theorem 2. (The Soundness Theorem: General case) Let S be any set of formulas. Then every theorem of S is a tautological consequence of S; i.e., for every formula A, if $S \vdash A$, then $S \models A$.

Proof: This uses similar ideas as the proof of the special case. We skip the proof.

Adequacy

Theorem 3. (The Adequacy Theorem for the formal system of Propositional Logic: Special Case) Every tautology is a theorem of Propositional Logic; i.e., for every formula A, if $\models A$, then $\vdash A$.

This theorem is more difficult to prove. We skip the proof until later.

Q: Can you guess the General Case for the Adequacy Theorem? (Coming up later in Chapter 3.)

Consistency

Q: Is there a formula A such that both A and $\neg A$ are theorems of P? Ans: No. Why? Because if there were such an A, then, by the Adequacy Theorem above, both A and $\neg A$ would be tautologies, which is impossible (why?).

Example 1. Let $S = \{A_1, A_2, A_3\}$, where $A_1 = (p \vee q)$, $A_2 = (\neg p \vee r)$, $A_3 = \neg(q \vee r)$.

Q: Find a formula B such that $S \vdash B$ and $S \vdash \neg B$.

Definition 1. Let S be a set of formulas. If $\exists B$ such that $S \vdash B$ and $S \vdash \neg B$, then we say S is **inconsistent**. If there is no such B, then we say S is **consistent**.

Q: Is the empty set consistent or inconsistent? Ans: This is the same as asking whether P is consistent or inconsistent, which we already answered above.

In homework, you'll prove: If S is satisfiable, then it's consistent. Hint for proof: Assume S is satisfiable and inconsistent, then get a contradiction.

Interesting fact: If S is inconsistent, then we can prove any formula from it! (Proof: Homework.)

Example 2. Let S be as in Example 1. Pick any formula B. Show $S \vdash B$.

Proof: Let $A = (q \vee r)$. We already showed that A and $\neg A$ can both be proved from S. So write *one* proof that contains both A and $\neg A$ in it (we don't care which comes last). Then, by using the Expansion Rule, we can add to our proof the formulas $(A \vee B)$ and $(\neg A \vee B)$. Finally, using Cut Rule, we derive B.

Decidability

Definition 2. A formal system is said to be **decidable** if there is an algorithm (a systematic method) for deciding whether any given formula has a proof or not.

Example 3. Is the system ADD from our first day decidable? Ans: Yes: given any formula A, count and compare the number of a's (or $|$'s) on the left with the number of b's (or $|$'s) on the right. A has a proof iff these two numbers are equal.

Q: Is P decidable? Ans: Yes, because A has a proof iff it's a tautology, and we know how to decide whether a formula is a tautology. (How?)

HW # 7, due Mon 19 Feb
Read Section 3.2. Preview Section 3.3.
Do: p. 88: 3(1), 4-6, 12(1,2).