Review

What is a truth function?

What does it mean for a given set of connectives to be adequate?

What does the Adequacy Theorem say?

What is Disjunctive Normal Form?

Given any truth table, how can we find a formula A which "gives" that truth table? Use the term *truth function* to restate this question more precisely.

Motivation

Recall that one of our main purposes in this course is to learn the limits of mathematics! In every branch of mathematics, we accept certain intuitive statements as axioms; then we try to prove other true statements from these axioms. So we wonder: have we picked enough axioms to prove every true statement, even those that we may think of a thousand years later?

We will see (much later) that in most branches of mathematics, there is no satisfactory (i.e., decidable) set of axioms that are enough for proving every true statement in that branch of mathematics—this is one of Gödel's incompleteness theorems.

We will also see, in this chapter, that in Propositional Logic, there *is* a satisfactory (i.e., decidable) set of axioms that are enough for proving every true statement in Propositional Logic (i.e., every tautology).

Definition of the formal system P

Language:

1. Symbols: $p_1, p_2, \cdots, \neg, \vee, (\ )$.
2. Formulas: F1: Each p_i is a formula. F2: If A, B are formulas, then so are $\neg A$ and $(A \vee B)$.

Remark. The connectives $\wedge, \rightarrow, \leftrightarrow$ can be written in terms of $\neg$ and $\vee$. Therefore we have not included them in the formal system, in order to make things as small as possible; this will make it easier to prove things about the formal system P.

Axioms: For every formula A, the formula $(A \vee \neg A)$ is an axiom.
Remark. Note that this is not just one axiom; it's really giving us infinitely many axioms, one for each formula. So it's called an *axiom scheme*. For example, $(p_1 \vee \neg p_1)$ is an axiom; $((p_1 \vee p_2) \vee \neg(p_1 \vee p_2))$ is also an axiom.

Rules of Inference:

Associative Rule: $\frac{(A\vee(B\vee C))}{((A\vee B)\vee C)}$ Contraction Rule: $\frac{(A\vee A)}{A}$

Expansion Rule: $\frac{A}{(A\vee B)}$ Cut Rule: $\frac{(A\vee B),(\neg A\vee C)}{(B\vee C)}$

Example 1. Here is a proof for the formula $p \vee (\neg p \vee q)$.

1. $\neg p \vee p$ (AXIOM). **2**. $(\neg p \vee p) \vee q$ (EXP). **3**. $\neg p \vee (p \vee q)$ (ASSOC).

Instead of starting with $\neg p \vee p$, we could have started with any formula in place of p; e.g., $\neg(p\vee r)\vee(p\vee r)$. We could also use any formula we desire in place of q. Therefore, we can prove any formula of the form $\neg A \vee (A \vee B)$, where A and B can be any formulas we like.

Even though the symbols $\wedge, \rightarrow, \leftrightarrow$ are not officially part of the formal system P, for convenience, we use them as abbreviations:

Definition 1. We agree on the following abbreviations: **1**. $\neg(\neg A \vee \neg B)$ is abbreviated by $A \wedge B$. **2**. $(\neg A \vee B)$ is abbreviated by $A \rightarrow B$. **3**. $(A \rightarrow B) \wedge (B \rightarrow A)$ is abbreviated by $A \leftrightarrow B$.

Example 2. We showed above that $\neg p \vee (p \vee q)$ is a theorem in P. How can we abbreviate this formula using the symbol $\rightarrow$? Ans: $p \rightarrow (p \vee q)$.

Example 3. Sometimes we are interested in proving a formula using some given formulas as hypothesis, in addition to the axioms and rules of inference. For example, given the formula $p \vee q$ as hypothesis, can we prove the formula $q \vee p$?

Yes: **1**. $p \vee q$ (HYP). **2**. $\neg p \vee p$ (AXIOM). **3**. $q \vee p$ (CUT).

Definition 2. Let S be a set of formulas. To give a **proof** of a formula B **using** the formulas in S means to write a list of formulas such that:

1. Each formula in the list is either an axiom or is in S or is obtained from the previous formulas in the list by rules of inference.
2. The last formula in the list is B.

We write: $S \vdash B$ (read: S proves B). If S happens to be the empty set, i.e., we are not using any formulas as hypothesis, then we write $\vdash B$, which says that B can be proved from just the axioms and rules of inference, without using any hypothesis. For emphasis, we sometimes write $\vdash_{\mathrm{P}}$ instead of $\vdash$.

Example 4. In the above example, what is S? What is B? Ans: $S = \{(p \vee q)\}$. $B = (q \vee p)$.

Derived rules of inference

Above, we showed that $\{(p \vee q)\} \vdash (q \vee p)$. As usual, p and q can be replaced by any formulas: $\{(A \vee B)\} \vdash (B \vee A)$.

So, in any list of formulas, if we have a formula of the form $(A \vee B)$, we can add to the list the formula $(B \vee A)$, by first writing the axiom $\neg A \vee A$ before it, and then using the CUT rule.

So we can think of this as a new rule of inference; it is not officially part of the formal system P, but, for convenience, we can pretend it is, and call it the Commutative Rule: $\frac{A \vee B}{B \vee A}$.

Our book introduces about ten other such "derived rules of inference." You should study them, and feel free to use them for homework, but do not need to memorize them.

Example 5. Let $B = \neg p \rightarrow (p \rightarrow q)$. Show that $\vdash B$. (Hint: First use a scratch paper to go backwards.)

Ans:

1. $\neg\neg p \vee \neg p$ (AXIOM). **2**. $(\neg\neg p \vee \neg p) \vee q$ (EXP). **3**. $\neg\neg p \vee (\neg p \vee q)$ (EXP). **4**. $\neg\neg p \vee (p \rightarrow q)$ (Abbrev). **5**. $\neg p \rightarrow (p \rightarrow q)$ (Abbrev).

HW # 6, due Mon 12 Feb
Read Section 3.1. Preview Section 3.2.
Do: p. 83: 6(1,2), 7(1,2), 8(1-4).