Review: Def of logical equiv ($\cong$) of two formulas.

Redundancy and Adequacy

Recall: Last time we talked about "unless." We (partially) created a truth table for it.

Q: How many rows did the table have? Why?

Q: How many possible 2-ary (binary, or 2-variable) truth tables are there, in general? Ans: There are four rows; the value in the last (third) column of the table in each of the four rows can be T or F. So there are $2^4 = 16$ possibilities.

Q: The table that corresponds to "unless" happens to be "constructible" using the familiar five connectives. Do you think all the 16 truth tables can be "constructed" using just the five connectives we have seen so far?

Q: Are any of the five connectives "redundant", i.e., can any of them be constructed using the others, so that we could always replace the symbol for that connective with a logically equivalent formula?

To answer these questions, we should first make them precise, by giving some definitions. Recall that an n-ary function is a function of n variables.

Definition 1. An n-ary **truth function** is a function from $\{\mathrm{T}, \mathrm{F}\}^n \to \{\mathrm{T}, \mathrm{F}\}$.

Example 1. The connective $\wedge$ defines a 2-ary truth function $H_\wedge : \{\mathrm{T}, \mathrm{F}\}^2 \to \{\mathrm{T}, \mathrm{F}\}$:

$$H_\wedge(\mathrm{T},\mathrm{T}) = \mathrm{T},\ H_\wedge(\mathrm{T},\mathrm{F}) = \mathrm{F},\ H_\wedge(\mathrm{F},\mathrm{T}) = \mathrm{F},\ H_\wedge(\mathrm{F},\mathrm{F}) = \mathrm{F}.$$

Do the same for $H_\neg$ (unary, or 1-ary) and $H_\to$ (binary, or 2-ary).

Each formula A with n propositional variables determines an n-ary truth function H_A

Example 2. Let $A = (p \wedge q) \vee r$. Find each of the following:

$H_A(\mathrm{T},\mathrm{T},\mathrm{F}) =?$ Ans: T.

$H_A(\mathrm{F},\mathrm{T},\mathrm{F}) =?$ Ans: F.

$H_A(\mathrm{F},\mathrm{F},\mathrm{T}) =?$ Ans: T.

Now we can say precisely what we mean by the question: "Are the five logic connectives enough to construct everything?"

Definition 2. A given set of connectives is said to be **adequate** iff for every truth function $G : \{\mathrm{T}, \mathrm{F}\}^n \to \{\mathrm{T}, \mathrm{F}\}$, there exists a formula A that uses only the given connectives, such that $H_A = G$.

(In other words, for any truth table you construct, there exists a formula A whose truth table is what you constructed.)

Example 3. Let G be the following function:

$$G(x, y, z) = \begin{cases} \mathrm{T} & \text{if } x = y = z \\ \mathrm{F} & \text{otherwise} \end{cases}$$

Construct the truth table that corresponds to G.

Find a formula A such that $H_A = G$. Ans: (there is more than one correct answer) $A = (p \leftrightarrow q) \wedge (q \leftrightarrow r)$.

Theorem 1. (Adequacy Theorem) The set of connectives $\{\neg, \vee\}$ is adequate. The set of connectives $\{\neg, \wedge\}$ is also adequate.

Instead of giving a formal proof, let's just illustrate the idea through examples.

[Construct a truth table with three variables, p_1, p_2, p_3, and pick arbitrary values for the fourth column. Let G be the truth function defined by this truth table. Then, using the following procedure, construct a formula A such that $H_A = G$.]

Step 1. For each row where the value of G is T, write the formula $(x_1 \wedge x_2 \wedge x_3)$, where x_i is either p_i or $\neg p_i$, according to whether in that row the value of p_i is T or F.

Step 2. Take the disjunction of the formulas constructed in the previous step.

Q: What if there are no rows in which G has a value of T? Ans: Just let $A = (p_1 \wedge \neg p_1) \vee \cdots \vee (p_3 \wedge \neg p_3)$.

The formula we obtain using this procedure is of the form $(* \wedge \cdots \wedge *) \vee \cdots \vee (* \wedge \cdots \wedge *)$, where each $*$ represents a propositional variable. A formula in this form is said to be in **disjunctive normal form** (DNF).

Q: The Adequacy Theorem states that just $\neg$ and $\vee$ are enough, that we shouldn't need $\wedge$ (or also "the other way around"). But the procedure gives us a formula that uses both $\wedge$ and $\vee$. How can we avoid one of them?

Ans: $(A \wedge B) \cong \neg(\neg A \vee \neg B)$. Why? Because $\neg(A \wedge B) \cong (\neg A \vee \neg B)$.

Remark. It follows from the Adequacy Theorem that the five connectives are redundant (we can get rid of three of them).

HW # 5, due Wed 7 Feb
Read Section 2.3. Preview Section 3.1.
Do: p. 72: 14, 15(1,2,3), 16, 18, 22(1). CH: 17, 31, 33.