
We will come back to the Compactness Theorem in Section 2.2 later.

Review defs: Truth assignment; tautology; satisfiable; tautological consequence.

Valid or invalid arguments

Example 1. Is each of the following arguments valid or invalid?

(a) If it rains and the fog settles in, then the flight will be canceled. It will rain. ∴ The flight will be canceled.

Ans: This argument has the form:

$$(r \wedge f) \rightarrow c \; ; \quad r \; ; \quad \therefore c.$$

To check its validity, we can check whether the formula $[((r \wedge f) \rightarrow c) \wedge r] \rightarrow c$ is a tautology. The answer turns out to be: No, it's an invalid argument.

(b) If it rains, then the flight will be canceled. It will not rain. ∴ The flight will not be canceled.

Ans: again No.

Logical equivalence

Example 2. Are these two formulas equivalent? $A = \neg(p \wedge q)$. $B = (\neg p \vee \neg q)$.

Ans: Yes. Why? Do a truth table for each, and show they are T or F at the same times. Or: show that $A \leftrightarrow B$ is a tautology.

Q: What does it mean for two formulas to be "equivalent?"

Definition 1. Two formulas A and B are said to be **logically equivalent** iff the formula $(A \leftrightarrow B)$ is a tautology.

Example 3. Are these two formulas equivalent? $C = \neg(p \wedge q) \rightarrow r$. $D = (\neg p \vee \neg q) \rightarrow r$.

Ans: Yes; in the previous example, we saw that A and B were taut equiv; so $A \rightarrow r$ and $B \rightarrow r$ are also taut equiv, because of the following theorem.

Theorem 1. (Replacement Theorem)

Suppose A and B are logically equivalent formulas, and C is a formula in which A appears. Then, if we replace A with B in C, we obtain a formula that is logically equivalent to C.

Proof: Skip. (See book if interested.)

Redundancy and Adequacy

Q: How do you write "p if q?" Ans: $q \rightarrow p$. How about "p only if q?" Ans: $p \rightarrow q$ ("only if" = "implies").

Q: Can you think of other English words that are used as logical connectives? Here are some: either...or; neither...nor; but; so; therefore; because; although; whenever; since; despite of; thanks to; as long as; while; assuming that; nevertheless; unless.

Q: How do you write "p unless q?" Ans: depends on the interpretation of "unless." Let's say it means: "p is true, except when q is true, in which case p is false." Then, we see that "p unless q" is the same as: $p \leftrightarrow \neg q$.

Q: Construct the truth table for "unless."

Q: If I had shown you this truth table without any introduction, would you have guessed that this table represents "unless?"

Q: How many possible 2-ary (binary, or 2-variable) truth tables are there? Ans: There are four rows; the value in the last (third) column of the table in each of the four rows can be T or F. So there are $2^4 = 16$ possibilities.

Q: The table that corresponds to "unless" happens to be "constructible" using $\neg$ and $\leftrightarrow$. Do you think all the 16 truth tables can be "constructed" using just the five connectives we have seen so far?

Q: Are any of the five connectives "redundant", i.e., can any of them be constructed using the others?

To answer these questions, we should first make them precise, by giving some definitions. Recall that an n-ary function is a function of n variables.

Definition 2. An n-**ary truth function** is a function from $\{\mathrm{T},\mathrm{F}\}^n \to \{\mathrm{T},\mathrm{F}\}$.

Example 4. The connective $\wedge$ defines a 2-ary truth function $H_\wedge : \{\mathrm{T},\mathrm{F}\}^2 \to \{\mathrm{T},\mathrm{F}\}$:

$$H_\wedge(\mathrm{T},\mathrm{T}) = \mathrm{T},\ H_\wedge(\mathrm{T},\mathrm{F}) = \mathrm{F},\ H_\wedge(\mathrm{F},\mathrm{T}) = \mathrm{F},\ H_\wedge(\mathrm{F},\mathrm{F}) = \mathrm{F}.$$

Do the same for $H_\neg$ (unary, or 1-ary) and $H_\to$ (binary, or 2-ary).

Each formula A with n propositional variables determines an n-ary truth function H_A

Example 5. Let $A = (p \wedge q) \vee r$. Find each of the following:

$H_A(\mathrm{T},\mathrm{T},\mathrm{F}) =$? Ans: T.

$H_A(\mathrm{F},\mathrm{T},\mathrm{F}) =$? Ans: F.

$H_A(\mathrm{F},\mathrm{F},\mathrm{T}) =$? Ans: T.

Now we can say precisely what we mean by the question: "Are the five logic connectives enough to construct everything?"

Definition 3. A given set of connectives is said to be **adequate** iff for every truth function $G : \{\mathrm{T},\mathrm{F}\}^n \to \{\mathrm{T},\mathrm{F}\}$, there exists a formula A that uses only the given connectives, such that $H_A = G$.

Example 6. Let G be the following function:

$$G(x,y,z) = \begin{cases} \mathrm{T} & \text{if } x = y = z \\ \mathrm{F} & \text{otherwise} \end{cases}$$

Find a formula A such that $H_A = G$.

Ans: (there is more than one correct answer) $A = (p \leftrightarrow q) \wedge (q \leftrightarrow r)$.

Theorem 2. (Adequacy Theorem) The set of connectives $\{\neg, \vee\}$ is adequate. The set of connectives $\{\neg, \wedge\}$ is also adequate.

Proof: Next time.

HW # 4, due Mon 5 Feb
Read Section 2.3. Do: p. 72: 4(1,2), 5(2a), 6(2a), 7.