Review vocabulary: Logical connectives; conjuction; disjunction; propositions; antecedent; conclusion; propsitional variable.

Review truth tables (from Math 140) for each of the five logical connectives.

Recall: The only time $p \to q$ is false is when? When p is T and q is F.

Remember def of when a proposition is said to be a tautology?

(Informal def:) A proposition is said to be a tautology if it is always true, regardless of whether each of its propisitional vars is T or F.

For more precise defs (and to learn new concepts such as satisfiability, tautological consequence, etc.) we first define truth functions and truth assignments.

Definition 1. A **truth assignment** is a function from the propostional variables $\{p_1, p_2, \cdots\}$ to $\{\mathrm{T}, \mathrm{F}\}$.

(In other words, we assign a value of T or F to each propositional variable.)

Example 1. Let ϕ be the truth assignment $\phi(p_n) = \begin{cases} \mathrm{T} & \text{if } n = 1, 2 \\ \mathrm{F} & \text{if } n > 2 \end{cases}$. Let $A = (p_1 \wedge p_3)$.

Q: Under the truth assignment ϕ, is A true or false? Ans: F. So we write $\phi(A) = \mathrm{F}$.

Definition 2. A formula A is said to be a **tautology** if for every truth assignment ϕ, $\phi(A) = \mathrm{T}$.

Definition 3. A formula A is said to be a **satisfiable** if for some truth assignment ϕ, $\phi(A) = \mathrm{T}$; we say ϕ **satisfies** A, or A is **satisfied** by ϕ.

Example 2. Determine whether each of the following formulas is a tautology, satisfiable, both, or neither.

1. $A = p_1 \vee \neg p_1$. 2. $B = p_1 \to p_2$.

Q: Give an example of an unsatisfiable formula.

Definition 4. Let S be a set of formulas. A truth assignment ϕ **satisfies** S if for every $A \in S$, $\phi(A) = \mathrm{T}$. S is said to be a **satisfiable** if there exists a truth assignment that satisfies S.

Example 3. (a) Let $A = p_2 \wedge p_3$, $B = p_1 \to p_2$. Let $S = \{A, B\}$. Prove S is satisfiable.

(b) Let $A = \neg p_2 \wedge p_1$, $B = p_1 \to p_2$. Let $S = \{A, B\}$. Is S satisfiable? Ans: No.

(c) Let $A_i = p_i \vee \neg p_{i+1}$. Let $S = \{A_1, A_2, \cdots\}$. Is S satisfiable? Ans: Yes, let $\phi(p_i) = \mathrm{T}$ for all i.

Definition 5. Let B be a formula, and S a set of formulas. We say B is a **tautological consequence** of S if every truth assignment that satisfies S also satisfies B; we write $S \models B$.

Example 4. Let $A_1 = p$, $A_2 = p \to q$, $S = \{A_1, A_2\}$, $B = q$. Show $S \models B$.

Proof. Let ϕ be any truth assignment that satisfies S. Then $\phi(p) = \mathrm{T}$. If q is false, then by definition, $H_{\to}(p, q) = H_{\to}(\mathrm{T}, \mathrm{F}) = \mathrm{F}$. But, by hypothesis, $\phi(p \to q) = \mathrm{T}$. So q cannot be false; it must be true. So $\phi(B) = \mathrm{T}$. □

Example 5. Let Let $S = \{p \to q\}$, $B = q$. Does $S \models B$? No. Why?

Algorithms for determining tautology, and tautological consequence

Can you think of an algorithm (a systematic method) for determining whether a formula is a tautology?

Example 6. Let $A = [p \wedge (p \rightarrow q)] \rightarrow q$. Is A a tautology? Yes; do a truth table for A; it will always come out true.

Can you think of an algorithm (a systematic method) for determining whether a formula B is a tautological consequence of a set of formulas $\{A_1, \cdots, A_n\}$?

Ans: Yes: let $C = (A_1 \wedge \cdots \wedge A_n) \rightarrow B$. Then use a truth table to check whether or not C is a tautology.

Q: Why does this trick work? Ans: We are using the following theorem:

Theorem 1. $(A_1 \wedge \cdots \wedge A_n) \rightarrow B$ is a tautology iff $\{A_1, \cdots, A_n\} \models B$

HW # 3, due Fri 2 Feb

Read Section 2.2. Do: p. 63: [1], 2-4, 20, 22, 23(1,2), 30(1), 32(2).