1. In Statement Calculus, suppose A and B are formulas, and Γ is a set of formulas. Prove or disprove each of the following.

   (a) If $\Gamma \vdash A \to B$ and $\Gamma \vdash A$, then $\Gamma \vdash B$.

   (b) If $\Gamma \vdash B$, then $\Gamma \vdash A \to B$.

   (c) If $\Gamma \vdash A \to B$ and $\Gamma \vdash B$, then $\Gamma \vdash A$. (Postponed till later.)

   (d) If $\vdash A$ and $\Gamma \vdash A \to B$, then $\Gamma \vdash B$.

2. In Statement Calculus, suppose A and B are formulas, and Γ is a set of formulas. Prove or disprove each of the following.

   (a) If $\Gamma \models A \to B$ and $\Gamma \models A$, then $\Gamma \models B$.

   (b) If $\Gamma \models B$, then $\Gamma \models A \to B$.

   (c) If $\Gamma \models A \to B$ and $\Gamma \models B$, then $\Gamma \models A$.

   (d) If $\models A$ and $\Gamma \models A \to B$, then $\Gamma \models B$.

3. Let $f(x) = x + 1$. In the following, all variables x, y, $\cdots$, range over the integers $\mathbb{Z}$.

   (a) Prove or disprove: For every x, $xf(x)$ is even.

   (b) Prove or disprove: For every x and y, $f(xyf(x))$ is odd.

   (c) Let's define an expression to be a **well-formed formula**, (wff for short), if it involves only multiplication and the function f, with correct use of parentheses. For example, $f(xyf(x))$ is a wff, but $x + yf(x)$ and $f)x$ and fx are not.

   Prove or disprove: Every wff is either always even or always odd.

4. **The FM Game.** (F for $f(x)$, M for multiplication; I couldn't think of a better name.)

   How the game is played: Your opponent gives you a wff, and you have to determine whether or not that wff can be obtained using the following rules.

FM1: For any wff A, you may write $f(A)A$.

FM2: If you've already written a wff A, then you may write BA for any wff B.

FM3: If you've already written a wff of the form AA, then you may write A.

FM4: If you've already written two wff's of the form AB and $f(A)C$, then you may write BC.

   Example: Here is a way to obtain the wff $yxf(y)$.

   1. $f(y)y$ FM1
   2. $f(y)yx$ 1, FM2
   3. $f(f(y))f(y)$ FM1
   4. $yxf(y)$ 2, 3, FM4

   Suppose your opponent gives you the wff $f(x)xf(x)x$. Prove that you can obtain it using FM1-4.

5. (a) Prove that any wff that can be obtained using FM1-4 is always even.

   (b) Do you think the converse to the above also holds?