
Examples of applications of matrix multiplication (not in book)

Example 1. We have two bags of trail-mix, each containing almonds and raisins. Bag 1: 90% almonds, 10% raisins. Bag 2: 20% almonds, 80% raisins. Let's denote this by the matrix $A = \begin{bmatrix} .9 & .1 \\ .2 & .8 \end{bmatrix}$. Suppose almonds contain 60% protein, 40% fat and carbohydrates, while raisins contain 0% protein, 100% fat and carbohydrates; Let's denote this by $B = \begin{bmatrix} .6 & .4 \\ 0 & 1 \end{bmatrix}$.

Q1: Compute AB . What does each entry of AB represent? Why?

Q2: Suppose we want to make a third bag containing almonds and raisin 50-50. What percentage of bag 3 should come from each of the two bags? Hint: Fill in the details, and give explanations: Let x_i represent the percentage of bag 3 that comes from bag i . Then: $\text{row}_1(A)x_1 + \text{row}_2(A)x_2 = [.5 \ .5]$; explain why. So $A^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = ?$ Why? So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = ?$ Why?

Example 2. Rotation Matrices. Let $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ be an arbitrary vector in \mathbb{R}^2 . Let $A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$, where α is some arbitrary angle in radians. Any 2×2 matrix that can be written in this form is called a **rotation matrix**.

Q1: Let $\vec{w} = A\vec{v}$. How is \vec{w} related to \vec{v} ? A: \vec{w} is obtained by rotating \vec{v} counterclockwise through an angle of α .

Proof. We can write x and y as: $x = r \cos(\theta)$, $y = r \sin(\theta)$. Then $A\vec{v} = \begin{bmatrix} \cos(\alpha)r \cos(\theta) - \sin(\alpha)r \sin(\theta) \\ \sin(\alpha)r \cos(\theta) + \cos(\alpha)r \sin(\theta) \end{bmatrix} = \begin{bmatrix} r \cos(\theta + \alpha) \\ r \sin(\theta + \alpha) \end{bmatrix}$, as desired.

□

Q2: Let $B = \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix}$, where β is also some arbitrary angle. Is AB a rotation matrix? A: Yes. Why?

Q3: What is A^{-1} ? Is it a rotation matrix of angle $-\alpha$? A: Yes. Why?

Example 3. (From *Elementary Linear Algebra*, by Spence, Insel, and Friedberg, page 108.) We have four cities with airports. Let A be the 4×4 matrix defined by

$$A_{i,j} = \begin{cases} 1 & \text{if there is a nonstop flight} \\ & \text{from city } i \text{ to city } j \\ 0 & \text{otherwise} \end{cases}$$

Q1: Suppose $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ Compute A^2 . What does $A^2_{2,3}$ represent? A: It represents the number of flights with exactly one layover between city 2 and city 3. Why?

(Can we do more? Does A^3 represent the flights with exactly two layovers? Why?)
