Definition 1. Let $\vec{v_1}, \cdots, \vec{v_n}$ be vectors. To say a vector $\vec{w}$ is a **linear combination** of $\vec{v_1}, \cdots, \vec{v_n}$ means there exist scalars $c_1, \cdots, c_n \in \mathbb{R}$ such that $c_1\vec{v}_1 + \cdots c_n\vec{v_n} = \vec{w}$. The numbers $c_1, \cdots, c_n$ are called **coefficients**.

Definition 2. A polynomial in several variables is said to be **linear** if the exponent of every variable is 1, and each variable is multiplied only by a constant, not by other variables.

Definition 3. A system is called **singular** if it has no solutions or infinitely many solutions. A system is called **nonsingular** if it has exactly one solution.

Definition 4. Two systems of linear equations are said to be **equivalent** if they have the same solutions (i.e., if any solution of one system is also a solution of the other).

Definition 5. A **row operation** is any of the following, where c is a nonzero scalar: 1. row$i$ = row$i$ + c(row$k$); 2. row$i$ = c(row$i$); 3. row$i$ = row$k$, and row$k$ = row$i$ (switch rows).

Definition 6. A **pivot** is the first non-zero entry in a row.

Definition 7. Let A be an $m \times n$ matrix (m rows, n columns). If $m = n$, then A is said to be a **square matrix**. For $1 \leq i \leq m$ and $1 \leq j \leq n$, the **(i, j)-entry** of A, denoted by $A_{i,j}$, is the number in the ith row and the jth column of A. We denote the ith row of A by $\mathbf{row}_i(A)$, and the jth column of A by $\mathbf{col}_j(A)$.

Definition 8. Let A and B be $m \times n$ matrices. Then their **sum** $A + B$ is an $m \times n$ matrix C defined by: $C_{i,j} = A_{i,j} + B_{i,j}$.

Definition 9. Let A be an $m \times n$ matrix, and B an $n \times q$ matrix. Then their **product** AB is an $m \times q$ matrix C defined by $C_{i,j} = \text{row}_i(A)\,\text{col}_j(B)$. (Equivalently, C can be defined by: $\text{col}_j(C) = A\,\text{col}_j(B)$.)

Definition 10. The $n \times n$ **identity matrix** I_n is defined to have 1's along its diagonal, and 0's elsewhere.

Definition 11. Two $n \times n$ matrices A and B are said to be **inverses** of each other if $AB = I_n$ and $BA = I_n$.

Definition 12. A matrix A that has an inverse is called **invertible**, and its inverse is denoted by A^{-1}. A matrix that does not have an inverse is called **non-invertible** or **singular**.

Definition 13. The **transpose** A^T of a matrix A is defined by: the ijth entry of A^T is the jith entry of A; i.e., $(A^T)_{i,j} = A_{j,i}$

Definition 14. A matrix A is called **symmetric** if $A^T = A$.

Definition 15. Let A be any matrix. The **(i, j)-minor** of A is the matrix obtained by removing its ith row and its jth column. It is denoted by $\hat{A}_{i,j}$. (Our book uses M_{ij}.)

Definition 16. Let A be any n by n matrix. The **determinant** of A is defined as: If $n = 1$, then $\det(A) = A_{11}$. If $n \geq 2$, then for any fixed row i, $\det(A) = \sum_{j=1}^{n}(-1)^{i+j}A_{i,j}\det(\hat{A}_{i,j})$; or, for any fixed column j, $\det(A) = \sum_{i=1}^{n}(-1)^{i+j}A_{i,j}\det(\hat{A}_{i,j})$.

Definition 17. Given a matrix A, the **cofactor** of its (i, j)-entry is defined as $C_{i,j} = (-1)^{i+j}\det(\hat{A}_{i,j})$.

Definition 18. Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^3$. Their **cross product** is defined as $\vec{u} \times \vec{v} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$.

Definition 19. The **column space** of an $m \times n$ matrix A is the set (i.e., the collection) of all vectors that are lin combs of the columns of A.

Definition 20. Let V be a set of vectors in $\mathbb{R}^n$ (for some n). V is said to be a **vector space** if it satisfies both of the following conditions: (1) It is **closed under vector addition**: for every $\vec{v}$ and $\vec{w}$ in V, $\vec{v} + \vec{w}$ is in V. (2) It is **closed under scalar multiplication**: for every $\vec{v}$ in V and for every scalar c, $c\vec{v}$ is in V.

Definition 21. If V and W are both vector spaces, and if $V \subset W$, then we say V is a **subspace** of W. ("$\subset$" is the subset symbol; it means: W contains all the vectors that are in V, plus more perhaps.)

Definition 22. Given any matrix A, the set of all vectors $\vec{x}$ that satisfy $A\vec{x} = \vec{0}$ is called the **nullspace** of A, denoted NS(A). (Book uses boldface N.)

Definition 23. The **trivial vector space** is the set $\{\vec{0}\}$ (it contains only the zero vector).

Definition 24. A matrix is in **echelon form** iff: 1. under every pivot the entries are all zeros; and 2. the pivots are in a "staircase" arrangement, i.e., if p is a pivot, then every pivot to the right of p is below p (more precisely, for any two pivots $A_{i,j}$ and $A_{k,l}$, $i < k$ iff $j < l$); and 3. all rows of zeros appear at the bottom, i.e., below all pivots.

Definition 25. A matrix is in **reduced row echelon form (rref)** iff: 1. it is in echelon form; and 2. the entries above each pivot are zero; and 3. all pivots equal 1.

Definition 26. The **rank** of a mtx A is the number of pivot columns in rref(A). The **nullity** of a mtx A is the number of free cols in rref(A).

Definition 27. In $A\vec{x} = \vec{b}$, if $\vec{b} = \vec{0}$, we say the system is **homogeneous**. Otherwise we say it is **nonhomogeneous**.

Definition 28. An $m \times n$ mtx A is said to have **full column rank** iff its rank equals its number of columns (r=n); i.e., there is a pivot in every col. A is said to have **full row rank** iff its rank equals its number of rows (r=m); i.e., there is a pivot in every row.

Definition 29. Let $\vec{v_1}, \cdots, \vec{v_n}$ be vectors in $\mathbb{R}^k$. Their **span** is defined as:

$\text{span}\{\vec{v_1}, \cdots, \vec{v_n}\} = \{\vec{w} \mid \vec{w} \text{ is a lin comb of } \vec{v_1}, \cdots, \vec{v_n}\}$ = the set of all vectors that are linear combinations of $\vec{v_1}, \cdots, \vec{v_n}$. If $V = \text{span}\{\vec{v_1}, \cdots, \vec{v_n}\}$, then we say $\vec{v_1}, \cdots, \vec{v_n}$ *span* V.

Definition 30. For any vector space V, the **dimension** of V, denoted $\dim(V)$, is the least number of vectors necessary to span V.

Definition 31. A set of vectors $\{\vec{v_1}, \cdots, \vec{v_n}\}$ is said to be **linearly dependent** iff at least one of them is equal to a linear combination of the others. A set of vectors $\{\vec{v_1}, \cdots, \vec{v_n}\}$ is said to be **linearly independent** iff none of them is equal to a linear combination of the others.

Equivalent Def: A set of vectors $\{\vec{v_1}, \cdots, \vec{v_n}\}$ is said to be **linearly dependent** if there exist scalars $c_1, \cdots, c_n$, at least one of which is nonzero, such that $c_1\vec{v_1} + \cdots + c_n\vec{v_n} = \vec{0}$. The vectors $\vec{v_1}, \cdots, \vec{v_n}$ are said to be **linearly independent** if the only time $c_1\vec{v_1} + \cdots + c_n\vec{v_n} = \vec{0}$ is when all the scalars c_i are 0.

Definition 32. Given a vector space V, a **basis** for V is a set of vectors $\vec{v_1}, \cdots, \vec{v_n}$ that (1) are linearly independent, and (2) span V.

Definition 33. The **row space** of a matrix A is defined as: $\text{RS}(A) = \text{span}(\text{rows of } A)$.

Definition 34. The **left nullspace** of an $m \times n$ mtx A is the set of all $1 \times m$ row vectors $\vec{y}$ such that $\vec{y}A = \vec{0}$. We write: $\text{LNS}(A) = \{\vec{y} \mid \vec{y}A = \vec{0}\}$. Equivalent definition: $\text{LNS}(A) = \text{NS}(A^T)$.

Definition 35. Given two vectors $\vec{v} = (v_1, \cdots, v_n)$ and $\vec{w} = (w_1, \cdots, w_n)$, their **dot product** (also called **inner product**) is defined to be: $\vec{v} \cdot \vec{w} = v_1 w_1 + \cdots + v_n w_n$.

Definition 36. Two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ are said to be **perpendicular** or **orthogonal** to each other iff $\vec{v} \cdot \vec{w} = 0$; we write $\vec{v} \perp \vec{w}$. The **length** or **norm** of $\vec{v}$ is defined as $||\vec{v}|| = \sqrt{{v_1}^2 + \cdots + {v_n}^2} = \sqrt{\vec{v} \cdot \vec{v}}$. A **unit vector** is a vector whose length is 1.

Definition 37. A vector $\vec{u}$ is said to be **perpendicular** or **orthogonal** to a vector space V iff $\vec{u}$ is perpendicular to *every* vector in V; we write $\vec{u} \perp V$.

Definition 38. The **orthogonal complement** of a vector space $V \subset \mathbb{R}^n$ is defined as $V^{\perp}$ = the set of all vectors in $\mathbb{R}^n$ that are orthogonal to V.

Definition 39. For $\vec{v} \in \mathbb{R}^n$ and W a subspace of $\mathbb{R}^n$, the **projection** of $\vec{v}$ onto W is defined as: $\text{proj}(\vec{v}, W)$ = a vector $\vec{p} \in W$ such that $\vec{v} - \vec{p}$ is orthogonal to W.

Definition 40. For $\vec{v}, \vec{w} \in \mathbb{R}^n$, the **projection** of $\vec{v}$ onto $\vec{w}$ is defined as: $\text{proj}(\vec{v}, \vec{w}) = \text{proj}(\vec{v}, L)$, where L is the line that contains $\vec{w}$.

Definition 41. A set of vectors $\{\vec{v_1}, \cdots, \vec{v_n}\}$ is said to be **orthonormal** if every $\vec{v_i}$ is a unit vector, and the vectors are all mutually perpendicular.

Definition 42. Let A be a square matrix. A nonzero vector $\vec{v}$ is an **eigenvector** for A if $A\vec{v}$ has the same direction as $\pm\vec{v}$, i.e., $A\vec{v} = \lambda\vec{v}$ for some scalar λ; λ is said to be the **eigenvalue** that corresponds to the eigenvector $\vec{v}$.

Definition 43. Given a square matrix A, $\det(A - \lambda I)$ is called the **characteristic polynomial** of A.

Definition 44. A matrix A is said to be **diagonal** if its nondiagonal entries are all zero, i.e., for $i \neq j$, $A_{i,j} = 0$.

Definition 45. A matrix A is said to be **diagonalizable** if there exists a matrix B such that $B^{-1}AB$ is a diagonal matrix.

Theorems

Theorem 1. Matrix multiplication is **associative**: $(AB)C = A(BC)$ (assuming "sizes match", so that the products are defined).

Theorem 2. Matrix multiplication is **distributive** with respect to addition: $A(B+C) = AB + AC$ (assuming "sizes match", so that the products are defined).

Theorem 3. If A and B are invertible matrices, then $(AB)^{-1} = B^{-1}A^{-1}$.

Theorem 4. If $ad - bc \neq 0$, then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Theorem 5. $(A+B)^T = A^T + B^T$. $(AB)^T = B^T A^T$.

Corollary 6. If A is invertible, then $(A^{-1})^T = (A^T)^{-1}$.

Theorem 7. A is invertible iff $\det(A) \neq 0$.

Theorem 8. $\det(AB) = \det(A)\det(B)$

Corollary 9. If A is invertible, then $\det(A^{-1}) = 1/\det(A)$.

Theorem 10. For any square matrix A, $\det(A^T) = \det(A)$.

Theorem 11. (1) *Constant Multiple Rule*: If B is obtained by multiplying one row or one column of a square matrix A by a constant c, then $\det(B) = c\det(A)$.

(2) *Switching Rule*: Switching any two rows of a matrix changes its det by a factor of -1. (Works also for nonadjacent rows.) Switching any two columns of a matrix changes its det by a factor of -1. (Works also for nonadjacent cols.)

(3) *Adding Rule*: Adding a multiple of one row to another (or of one column to another) does not change the det.

Theorem 12. The area of the parallelogram with sides $[a \;\; b]$ and $[c \;\; d]$ is given by the absolute value of the det of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Theorem 13. Given three vectors $[a \;\; b \;\; c]$, $[d \;\; e \;\; f]$ and $[g \;\; h \;\; k]$, the area of the parallelepiped in $\mathbb{R}^3$ with these vectors as its sides is equal to the absolute value of the det of $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$.

Theorem 14. For any $m\times n$ mtx A, $\text{rank}(A) = \text{rank}(A^T)$.

Theorem 15. Every vector space has a basis. All bases of a vector space have the same number of vectors in them.

Theorem 16. In an n-dimensional vector space, any set with more than n vectors is lin dep.

Theorem 17. For every matrix A, each of $\text{CS}(A)$, $\text{RS}(A)$, $\text{NS}(A)$, and $\text{LNS}(A)$ is a vector space.

Theorem 18. Fundamental Theorem of Linear Algebra, Part I (FTLA): For any $m \times n$ mtx A, $\dim(\text{RS}(A)) + \dim(\text{NS}(A)) = n$; $\dim(\text{CS}(A)) + \dim(\text{LNS}(A)) = m$.

Theorem 19. Column operations do not change the column space of a matrix. Row operations do not change the row space of a matrix. Row operations do not change the nullspace of a matrix.

Theorem 20. Row operations on a matrix do not change the lin dep/indep of the cols. Col operations on a matrix do not change the lin dep/indep of the rows.

Theorem 21. For any $m\times n$ mtx A, the pivot cols form a basis for $\text{CS}(A)$. So $\dim(\text{CS}(A)) = \text{rank}(A)$.

Note. By pivot cols of A we mean those columns that, after Gauss-Jordan elimination, *become* pivot cols in $\text{rref}(A)$.

Theorem 22. For any $m \times n$ mtx A, the special sols to $A\vec{x} = \vec{0}$ form a basis for $\text{NS}(A)$. Therefore $\dim(\text{NS}(A)) = \text{nullity}(A)$.

Theorem 23. For any matrix A, $\dim(\text{CS}(A)) = \dim(\text{RS}(A))$. In other words, for *any* mtx A, # of lin indep cols = # of lin indep rows!

Theorem 24. (FTLA, part 2): For any $m\times n$ mtx A, its row space and nullspace are orthogonal complements of each other in $\mathbb{R}^n$; and its col space and left nullspace are orthogonal complements of each other in $\mathbb{R}^m$.

Theorem 25. (1) $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$; (2) $\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$; (3) $c(\vec{v}\cdot\vec{w}) = (c\vec{v})\cdot\vec{w}$.

Theorem 26. Given any two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$, $\vec{v}\cdot\vec{w} = ||\vec{v}||\cdot||\vec{w}||\cos(\theta)$, where θ is the angle between the two vectors.

Theorem 27. $\text{proj}(\vec{v},\vec{w}) = (\vec{v}\cdot\frac{\vec{w}}{||\vec{w}||})\frac{\vec{w}}{||\vec{w}||}$

Theorem 28. For any square mtx A, the following statements are equivalent (i.e., they imply each other): (1) A is invertible; (2) $\det(A) \neq 0$; (3) A has full col rank; (4) A has full row rank; (5) the cols of A are lin indep; (6) the rows of A are lin indep.

Theorem 29. Every orthonormal set is linearly independent.

Theorem 30. In an n-dimensional vector space W, a set of n vectors is lin indep iff it spans W.

Theorem 31. Suppose A is an $n\times n$ matrix with n linearly independent eigenvectors $\vec{v_1},\cdots,\vec{v_n}$. Let $\lambda_1,\cdots,\lambda_n$ be the corresponding eigenvalues (not necessarily distinct). Let Λ be the diagonal matrix with $\Lambda_{i,i} = \lambda_i$. Let S be the matrix whose jth column is the eigenvector $\vec{v_j}$. Then $AS = S\Lambda$; so $S^{-1}AS = \Lambda$.

Theorem 32. (Converse of previous theorem.) Let A and B be $n\times n$ matrices such that $B^{-1}AB$ is a diagonal matrix D. Then each column of B is an eigenvector of A, with corresponding eigenvalues appearing on the diagonal of D.

Theorem 33. An $n\times n$ mtx is diagonalizable iff it has n lin indep e-vecs.