Recall def of diagonal matrix: both upper- and lower-triangular.

Definition 1. A matrix A is said to be **diagonal** if its nondiagonal entries are all zero, i.e., for $i \neq j$, $A_{i,j} = 0$.

(The diagonal entries may or may not be zero.)

Diagonal matrices are "easy"

Example 1. It is easy and quick to tell whether a diagonal mtx is invertible or not; how? Ans: its det is the product of its diagonal entries.

Example 2. Let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$. Find A^n for any natural number n. Ans: $A^n = \begin{bmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{bmatrix}$.

Example 3. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$. From before, we get e-vals $\lambda_1 = 2$, $\lambda_2 = 3$, e-vecs $\vec{v_1} = (2, -1)$, $\vec{v_2} = (1, 1)$.

Q: Let S be the matrix whose jth column is the eigenvector $\vec{v_j}$. Write out what S is.

Q: Compute $S^{-1}AS$. Ans: it's a diagonal mtx Λ with $\Lambda_{1,1} = \lambda_1 = 2$, and $\Lambda_{2,2} = \lambda_2 = 3$.

This is not a coincidence!

Theorem 1. Suppose A is an $n \times n$ matrix with n linearly independent eigenvectors $\vec{v_1}, \cdots, \vec{v_n}$. Let $\lambda_1, \cdots, \lambda_n$ be the corresponding eigenvalues (not necessarily distinct). Let Λ be the diagonal matrix with $\Lambda_{i,i} = \lambda_i$. Let S be the matrix whose jth column is the eigenvector $\vec{v_j}$. Then $S^{-1}AS = \Lambda$.

Proof. $AS = [A\mathrm{col}_1(S) \ \cdots \ A\mathrm{col}_n(S)] = [A\vec{v_1} \ \cdots \ A\vec{v_n}] = [\lambda_1\vec{v_1} \ \cdots \ \lambda_n\vec{v_n}] = S\Lambda$. Since $\vec{v_1}, \cdots, \vec{v_n}$ are lin indep, by a previous thm S is invertible; so $S^{-1}AS = \Lambda$. □

Theorem 2. (Converse of previous theorem.) Let A and B be $n \times n$ matrices such that $B^{-1}AB$ is a diagonal matrix D. Then each column of B is an eigenvector of A, with corresponding eigenvalues appearing on the diagonal of D.

Proof: On practice final.

Definition 2. A matrix A is said to be **diagonalizable** if there exists a matrix B such that $B^{-1}AB$ is a diagonal matrix.

Example 4. The matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ from the example above is diagonalizable. Why?

Corollary 3. An $n \times n$ mtx is diagonalizable iff it has n lin indep e-vecs.

Example 5. Prove that if $BAB^{-1} = D$, then $A^n = B^{-1}D^nB$. Ans: $A = B^{-1}DB$.

So $A^n = (B^{-1}DB)(B^{-1}DB)\cdots(B^{-1}DB)$ (multiply n times). Simplify: $A^n = BDID\cdots IDB^{-1} = BD^nB^{-1}$.

Example 6. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$. Use the e-vec mtx S and the e-val mtx Λ and the above example to find A^{100}.

Example 7. Fibonacci numbers: $0, 1, 1, 2, 3, 5, 8, 13, 21, \cdots$. Let $f(n) =$ the nth Fibonacci number. Then $f(0) = 0$, $f(1) = 1$; $f(n) + f(n+1) = f(n+2)$. This is a *recursive* formula: we use previous values of f to find the next one.

Find a formula for $f(n)$ in terms of n only (a non-recursive formula).

"Trick": $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} f(n) \\ f(n+1) \end{bmatrix} = \begin{bmatrix} f(n+1) \\ f(n+2) \end{bmatrix}$. (Why?)

So $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} f(0) \\ f(1) \end{bmatrix} = \begin{bmatrix} f(n) \\ f(n+1) \end{bmatrix}$. (Why?)

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Then $A^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} f(n) \\ f(n+1) \end{bmatrix}$.

Find e-vals and e-vecs: $\lambda_1 = (1+\sqrt{5})/2$ (the Golden Ratio!), $\lambda_2 = (1-\sqrt{5})/2$. $\vec{v_1} = ((-1+\sqrt{5})/2, 1)$, $\vec{v_2} = ((-1-\sqrt{5})/2, 1)$.

Compute A^n as in the previous example. (This is a bit of work; for a nice shortcut, see the book.)

Then we get the amazing formula:

$$f(n) = \frac{1}{\sqrt{5}} \left[\left(\frac{(1+\sqrt{5})}{2} \right)^n - \left(\frac{(1-\sqrt{5})}{2} \right)^n \right]$$