Motivation

Multiplication of an $m \times n$ matrix A by a vector $\vec{x}$ can be thought of as a function from $\mathbb{R}^n$ to $\mathbb{R}^m$: $f(\vec{x}) = A\vec{x}$. (Draw diagram.) Instead of concentrating on solving $A\vec{x} = \vec{b}$, we now shift our view a little in order to understand this function.

Let's look only at the case when A is a square matrix: $n \times n$. Then A is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Let's say $n = 2$. What does A do to the xy-plane? Some matrices rotate the plane. Some stretch or contract it. Some collapse it to a line, or to a point.

Example 1. Describe (using pictures and words) what each of the following matrices does to $\mathbb{R}^2$.

(a) $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Ans: doesn't change direction of any vector; makes every vector twice as long.

(b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Ans: stretches the plane vertically be a factor of two. The x- and y-axes don't change direction. But all other lines do change direction (they become steeper).

(c) $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Ans: x-axis turns counter-clockwise 45 degrees, and stretches by a factor of $\sqrt{2}$; y-axis collapses to a point.

(d) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Ans: x-axis turns counter-clockwise by $\arctan(1/2)$, and stretches by a factor of $\sqrt{5}$; y-axis turns clockwise by $\arctan(1/2)$, and stretches by a factor of $\sqrt{5}$.

Q: For the last matrix, are there any vectors that do not change direction? Ans: Yes: $(1, 1)$.

Definition 1. Let A be a square matrix. A nonzero vector $\vec{v}$ is an **eigenvector** for A if $A\vec{v}$ has the same direction as $\pm\vec{v}$, i.e., $A\vec{v} = \lambda\vec{v}$ for some scalar λ; λ is said to be the **eigenvalue** that corresponds to the eigenvector $\vec{v}$.

(? "Eigen" = own, peculiar, characteristic)

Example 2. Find an eigenvector and an eigenvalue for $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Ans: eigenvector: $(1, 1)$; corresponding eigenvalue: 3.

Example 3. Is there a matrix that has no eigenvectors (i.e., every vector does change direction)? Ans: Yes, a matrix that rotates the whole plane: Pick any angle, say $\theta = 30°$. Let $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

Q: Let $\vec{e_1} = (1, 0)$, $\vec{e_2} = (0, 1)$. Compute $A\vec{e_1}$ and $A\vec{e_2}$.

How to find eigenvectors and eigenvalues

Example 4. Find eigenvectors and eigenvalues for $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ by following the steps below.

We want $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$, for some scalar λ. So $A\vec{x} - \lambda\vec{x} = \vec{0}$, so $A\vec{x} - \lambda I\vec{x} = \vec{0}$, so $(A - \lambda I)\vec{x} = \vec{0}$, so $\vec{x}$ is in the nullspace of $A - \lambda I$.

Q: We want a *nonzero* sol; why? Ans: By def, $\vec{0}$ is not considered an e-vec.

Q: Should $\det(A - \lambda I)$ be zero or nonzero for $(A - \lambda I)\vec{x} = \vec{0}$ to have a nonzero solution? Ans: 0.

Q: For what values of λ is $\det(A - \lambda I) = 0$?

Ans: $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix}$. So $\det(A - \lambda I) = (1-\lambda)(4-\lambda) - (-1)(2) = \lambda^2 - 5\lambda + 6$. This is called the *characteristic polynomial* for the matrix A.

We want to find the roots of the polynomial. Why?

The roots are: $\lambda_1 = 2,\ \lambda_2 = 3$. These are the e-vals for A.

Q: How do we find the corresponding eigenvectors? Ans: For each e-val λ, compute the matrix $A - \lambda I$, and find solutions to $(A - \lambda I)\vec{x} = \vec{0}$.

We get eigenvectors: for λ_1, $\vec{v_1} = (2, -1)$; for λ_2, $\vec{v_2} = (1, 1)$.

Definition 2. Given a square matrix A, $\det(A - \lambda I)$ is called the **characteristic polynomial** of A.

Outline of process for finding eigenvectors and eigenvalues

Step 1. Find roots of characteristic polynomial ($\det(A - \lambda I) = 0$) . These are the e-vals.

Step 2. For each root λ_i, find the special solutions to $(A - \lambda_i I)\vec{x} = \vec{0}$. These are the e-vecs.

Note. Sometimes there are no e-vals: the characteristic polynomial may have no real roots (it might have complex roots, which we don't work with here). In that case there are no e-vecs!

Example 5. Find all e-vals and e-vecs for $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

Step 1. The characteristic polynomial is: $\det \begin{bmatrix} 0-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = -\lambda(1-\lambda) - 1 = \lambda^2 - \lambda - 1$.

Use the quadratic formula to find the roots: $\lambda_1 = (1+\sqrt{5})/2$ (the Golden Ratio!), $\lambda_2 = (1-\sqrt{5})/2$.

Step 2. Find e-vecs for λ_1: $\begin{array}{c} -\lambda_1 x + y = 0 \\ x + (1-\lambda_1)y = 0 \end{array}$. The two equations are multiples of each other: $(1-\lambda_1)(\text{eqn1}) = \text{eqn2}$, b/c, by above, $-\lambda(1-\lambda) = 1$.

Pick y as a free variable: $y = 1$, so, by eqn2, $x = \lambda_1 - 1 = (-1+\sqrt{5})/2$. So our first e-vec is: $\vec{v_1} = ((-1+\sqrt{5})/2, 1)$.

Similarly, we find our second e-vec, corresponding to λ_2, to be: $\vec{v_2} = ((-1-\sqrt{5})/2, 1)$.

Linear Algebra has many applications, including: solving systems of differential eqns (sec 6.3); fitting curves to data (least squares, sec 4.3); computer graphics (ch 8); linear programming; Graph Theory (ch 8); Economics (ch 8); physics; biology; chemistry; most sciences; etc.

Next time we'll see an application with Fibonacci Numbers.