Example 1. Q: Give a basis for $\mathbb{R}^2$. Q: Now give another basis.

Q: Can you give a basis for $\mathbb{R}^2$ such that each vector is a unit vector, and all the vectors are mutually perp? Ans: $(1,0)$, $(0,1)$.

Q: Can you find another such basis? Ans: $(1,2)/\sqrt{5}$, $(2,-1)/\sqrt{5}$.

Such a set of vectors is called *orthonormal*:

Definition 1. A set of vectors $\{\vec{v_1},\cdots,\vec{v_n}\}$ is said to be **orthonormal** if every $\vec{v_i}$ is a unit vector, and the vectors are all mutually perpendicular.

Example 2. Q: Find an orthonormal basis for $\mathbb{R}^3$. Ans: $\{(1,0,0),(0,1,0),(0,0,1)\}$.

Q: Is $\{(1,0,0),(0,1,0),(0,0,2)\}$ an orthonormal set of vectors? Ans: No, the last vector is not a unit vector. (This is only an *orthogonal* set; but it's not "normalized").

Q: Is $\{(1/\sqrt{2},1/\sqrt{2},0),(0,1,0)\}$ an orthonormal set of vectors? Ans: No, the two vectors are not perp.

Q: Is $\{(1/\sqrt{2},1/\sqrt{2},0),(1/\sqrt{2},-1/\sqrt{2},0)\}$ an orthonormal set of vectors? Yes.

Q: Can you find another orthonormal basis for $\mathbb{R}^3$? Ans: take any two unit vectors $\vec{u},\vec{v}\in\mathbb{R}^3$ that are in the xy-plane and are perp to each other; then $\{\vec{u},\vec{v},(0,0,1)\}$ is an orthonormal basis for $\mathbb{R}^3$. Why is it a basis? Because of the following theorems.

Theorem 1. Every orthonormal set is linearly independent.

Theorem 2. In an n-dimensional vector space W, a set of n vectors is lin indep iff it spans W.

How to find the projection of a vector onto a plane (or a hyperplane)

Example 3. Find an orthonormal basis for the plane $x-2y+z=0$. Ans: First find a basis, then make it orthonormal:

Step 1. Find a basis: the special solutions for $x-2y+z=0$ (or any two lin indep vectors in the plane): $(2,1,0),(-1,0,1)$.

Step 2. Use the Gram-Schmidt process (explained in more detail in the Gram-Schmidt handout) to make the basis orthonormal:

Let $\vec{v_1}=(2,1,0),\vec{v_2}=(-1,0,1)$.
"Make" $\vec{v_1}$ a unit vector: $\vec{u_1}=\vec{v_1}/||\vec{v_1}||$;
"Make" $\vec{v_2}$ a unit vector that's perp to $\vec{u_1}$ by: $\vec{u_2}=\dfrac{\vec{v_2}-\text{proj}(\vec{v_2},\vec{u_1})}{||\vec{v_2}-\text{proj}(\vec{v_2},\vec{u_1})||}$

Example 4. Let $\vec{b}=(2,3,5)$.

Q: Is $\vec{b}$ on the plane $x-2y+z=0$? No. Why?

Q: Find the closest vector to $\vec{b}$ in the plane V defined by $x-2y+z=0$; i.e., find $\text{proj}(\vec{b},V)$.

Ans: Use orthonormal basis found above: $\text{proj}(\vec{b},V)=(\vec{b}\cdot\vec{u_1})\vec{u_1}+(\vec{b}\cdot\vec{u_2})\vec{u_2}$.

Example 5. Suppose we're given an equation $A\vec{x}=\vec{b}$ such that $\vec{b}\notin\text{CS}(A)$. Then $A\vec{x}=\vec{b}$ has no solution; why?

Since there's no solution, we might be interested in the next best thing: find an $\vec{x}$ such that $A\vec{x}$ is as close as possible to $\vec{b}$. For any $\vec{x}$, the vector $A\vec{x}$ is in CS(A); why? So we're looking for a vector in CS(A) that's as close as possible to $\vec{b}$; why? And this is exactly what $\text{proj}(\vec{b}, \text{CS}(A))$ is!

Once we compute what $\text{proj}(\vec{b}, \text{CS}(A))$ is, we can find our desired $\vec{x}$, because the equation $A\vec{x} = \text{proj}(\vec{b}, \text{CS}(A))$ is guaranteed to have a solution; why?

Do this for $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\vec{b} = (2, 3, 5)$. Then $A\vec{x} = \vec{b}$ has no solution; why? Find the closest vector $\vec{b'}$ to $\vec{b}$ for which $A\vec{x} = \vec{b'}$ has a solution.

Ans: $\vec{b'} = \text{proj}(\vec{b}, \text{CS}(A))$, which we calculated above.

Example 6. Start with any three lin indep vectors in $\mathbb{R}^3$ (or $\mathbb{R}^4$). Apply the Gram-Schmidt Process (see handout) to obtain an orthonormal set of vectors with the same span. ...

Theorem 3. Any set of lin indep vectors can be extended to a basis. More precisely, let W be an n-dimensional vector space, and let $\vec{v_1}, \cdots, \vec{v_k}$ be lin indep, where $k < n$. Then one can always find vectors $\vec{v_{k+1}}, \cdots, \vec{v_n}$ such that $\{\vec{v_1}, \cdots, \vec{v_n}\}$ is a basis for W.

Seeing FTLA a little better (Optional)

Let A be an $m \times n$ mtx. Let $r = \text{rank}(A)$. Then $\dim(\text{RS}(A)) = \dim(\text{CS}(A)) = r$.

Pick an orthonormal basis B for RS(A). Pick an orthonormal basis C for NS(A). By FTLA, RS and NS are orthogonal complements, so every vector in B is perp to every vector in C.

It's then easy to prove (challenge problem) that $B \cup C$ is lin indep. In fact, it's an orthonormal basis for $\mathbb{R}^n$.

So every vector $\vec{v} \in \mathbb{R}^n$ can be written as a-vector-in-RS + a-vector-in-NS: $\vec{v} = \vec{v}_R + \vec{v}_N$.

Then $A\vec{v} = A\vec{v}_R + A\vec{v}_N = A\vec{v}_R + \vec{0}$.

$A\vec{v}_R$ is a vector in CS(A). So for any vector $\vec{v} \in \mathbb{R}^n$, A takes it to the same vector as $A\vec{v}_R$.

In other words, all the "action" of A is concentrated in taking vectors in RS to vectors in CS. Each of these is an r-dimensional vector space, and A is a bijection between these two vector spaces.
